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ABSTRACT

This is a manual for teachers using School Mathematics Study Group (SMSG) primary school text materials. The commentary is organized into three parts. The first part contains an introduction and a section on mathematical comments relating to background knowledge needed by the teacher. The second and main part consists of a chapter by chapter commentary of the text. The objectives, vocabulary, materials, and suggested procedures are given for each lesson. The third part suggests probability devices which might be of use. Chapter topics include: (1) certainty and uncertainty; (2) likelihood; (3) combining events; (4) possibilities; (5) combinations; (6) arrangements; (7) probability; and (8) repeated trials. (MP)

PROBABILITY FOR PRIMARY GRADES

Teacher's Commentary

(Revised Edition)

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PROBABILITY UNIT - PRIMARY GRADES

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PROBABILITY FOR PRIMARY GRADES

Introduction

Why Do We Teach Probability?

The study of probability has come to be an important part of the mathematics curriculum on higher levels. Probability has many practical uses. It is used in making decisions in the fields of military operations, scientific research, designing and manufacturing machines, insurance calculations, business predictions, weather forecasting, and in many other areas.

Probability is, of course, also used in games of chance. In fact, early research in the study of probability developed from a mathematician's interest in a friend's unsuccessful and imprudent wager.

"Goals for School Mathematics", the report of the 1963 Cambridge Conference, recommended that some of the basic ideas of probability should be introduced very early in the school program, and the School Mathematics Study Group undertook the development of materials for grades kindergarten to high school.

What Are We Trying To Do In This Unit?

Even quite young children are not strangers to probability on a practical level. They hear the words "not very likely", and they play games using spinners or dice, or games which involve drawing a card from a shuffled deck. By the very fact of playing the game, children demonstrate their idea that they have a chance of winning equal to that of the opponent. Take advantage of their offers to bring game materials from home, and discuss with the class any devices brought in.

It is on this level, and with these primitive ideas of "equal chance", "better chance", "not as likely as", "sure thing", etc., that the unit on probability for primary grades is written. To use and improve children's intuitive knowledge of what constitutes "an equal chance" or "a better chance", and to show -- though not in so many words -- that there is mathematical justification for many of their ideas will, it is hoped, make further study of probability easier, more desirable, and more fruitful.

How Does the Unit Fit Into the General Mathematics Program for the Primary Grades?

In addition to the fact that probability is an important part of mathematics, there are certain fringe benefits to be derived from the unit.

First, it should be fun for the children, and making mathematics fun is one way to improve instruction. Perhaps, because games are used as an integral part of the development of the unit, some children who have heretofore not enjoyed mathematics or who have found the work too hard will become more interested or will experience an unusual success.

Second, one of the goals of mathematics instruction is to promote systematic thinking rather than a hit-or-miss approach or jumping to conclusions. Especially the lessons on combinations and permutations (a word not used in the unit, by the way) demonstrate the advantages of a systematic approach and provide many opportunities for practice.

Third, certain arithmetical skills may be practiced and reinforced. Addition facts are used in many lessons, and subtraction and multiplication are required in some of the supplementary material. For those classes in which practice in the use of rational numbers (comparison and addition) is needed, there are opportunities galore. Children who have had difficulty in forming or interpreting addition and multiplication tables may find them easier to use after this unit.

Fourth, there are opportunities for independent investigation -- one might almost say "research" -- by individual children. The teacher, too, may seize opportunities to undertake more difficult concepts if it seems desirable with some groups of children.

How Shall I Use This Unit?

Because it is assumed that there has been no formal instruction in probability in the primary grades, the unit is not labeled "grade one" or "grade three", for example. The teacher should, preferably, look over the material in the entire unit, which has been written in sequential form. The decisions as to where to start, how much emphasis or time to give any lesson, what to omit, and how far or how fast to go are left to the teacher's judgment. Class interest and ability will, of course, be determining factors.

This unit consists of eleven lessons. Experience indicates that Lessons 1-5 can be understood by children in kindergarten and grade one while second grade

children seem able to comprehend quite well through Lesson 8. Pupils in third grade should be able to complete the unit. However, it is difficult to assign lessons to specific grade levels because of the wide variations among and within groups of children. The teacher can best determine how this unit can be used. It might be more profitable to incorporate it with the present mathematics program throughout the year instead of completing the unit in a short period of time.

The objective, vocabulary, materials, and suggested procedure are given for each lesson. The procedure is intended to be illustrative and not prescriptive.

It would be worthwhile to study the SMSG unit, Probability for the Intermediate Grades, to see the development in these grades. Also the junior high units, Introduction to Probability, Parts 1 and 2, will be most helpful. These two volumes contain a more sophisticated treatment of probability.

What Materials Are Needed?

Much of the material the child will encounter in later studies of probability will deal extensively with the results of coin tossing or dice throwing. In this introductory unit, coins and dice have not been used in the pupil text. The materials necessary for each lesson are listed. These include spinners which are provided with this unit. The regions of these spinners, by the way, are numbered so that they can be used in many ways for practice of the basic facts in arithmetic. For example, two spinners can be spun and the resulting numbers added. Each of the three types of spinners has a different set of numbers. Some activities employing these numbers have been outlined in the commentary, but the teacher is encouraged to use her imagination in the development of others. Colored blocks and marbles and bags to hold them, crayons, colored chalk, construction paper, paper cups, and tagboard are also used.

A section at the end of this Teachers' Commentary describes various unusual devices about which children should have no preconceptions and which, it is hoped, will stimulate curiosity and interest. Some children can probably make or help to make many of them. Some of these materials can be used to supplement lessons; others may be used for individual work. Feel free to substitute one device, color, etc., for another as is convenient.

One last point: Children should not be expected to read independently the instructions or questions on the pupil pages. It is assumed that these will be read aloud and more detailed instructions given if necessary.

1. The idea of probability.

Some events are regarded as certain: the sun will rise tomorrow, the coin will fall either heads or tails.

Other events are uncertain, depending upon chance fluctuations: it will rain tomorrow, the coin will fall heads.

In everyday conversation, people frequently compare the likelihood of (uncertain) events:

> They are all equally likely.
>
> This is more likely than that.
>
> I think it will rain this afternoon. (It is more likely to rain this afternoon than not to rain.)

In more scientific work, we compare likelihoods numerically. Every event is assigned a number describing or "measuring" its likelihood; the more likely the event, the higher the number assigned to it. The numerical measure of the likelihood of an event is called the probability of the event. The method of assigning probabilities to events and the study of the relations among them constitute the Theory of Probability.

A comment may be in order about the word "theory." In mathematics, the word is used to mean "body of knowledge." This is in contrast with its conversational meaning of "conjecture," as when we say, "The detective had a theory about who fired the gun." The theory of probability consists of our accumulated knowledge about probability. Similarly, mathematicians speak of the theory of numbers, the theory of groups, the theory of functions of a real variable, etc. No speculation is involved. Everything in a mathematical theory is true.

Modern science, industry, agriculture, and human affairs all depend strongly upon the theory of probability, either directly or via the theory of mathematical statistics which itself is based upon probability. The theory of probability underlies the biological laws of heredity; it is needed in fundamental research in physics and astronomy and, as a matter of fact, in working out plans for travel into space; it is used in developing and testing new drugs and medicines; it is the basis of agricultural experiments that lead to

improved methods for the farmer; it guides the manufacturer in controlling the quality of his product, the industrial laboratory in the design of new equipment, the economist and psychologist in their studies of behavior, the military commander in his choice of tactics.

Probability theory is also the basis for analyzing games of chance. In fact, the theory of probability had its origins, about 300 years ago, in the gambling halls of Europe. Moreover, in explaining the theory and in working problems, it is necessary to toss coins and throw dice. Occasionally a person will conclude that to teach probability is immoral. He reasons: "Gambling is evil. Gamblers have to use probability. Therefore we should not teach probability." One might equally well reason: "Gambling is evil. Gamblers have to read and write. Therefore we should not teach reading or writing."

We teach probability not because of its possible misuse but because of its applications for the benefit of mankind.

A particularly important application is to mass behavior where each individual action is subject to chance fluctuations. For example, while the theory of probability will not tell us whether the next coin tossed will actually land heads, it does tell us that of the next 10,000 tosses the number of heads will lie, with virtual certainty, between 4500 and 5500. Despite the randomness, there is stability in the whole. The principle applies (with perhaps different numerical values) not only in genetics and the other pure sciences but to industrial and social phenomena as well: to manufactured items rolling off an assembly line, to a mass of vacationers deciding where to stop for lunch.

2. <u>How probabilities are found</u>.

The probability assigned to any event is, roughly speaking, the fraction of the time we expect it to happen. The probability of an event is, therefore, a number between 0 and 1, inclusive. An <u>impossible</u> event is assigned the probability 0. (It never happens.) A <u>certain</u> event is assigned the probability 1 ($= 100\ \%$). An event that is just as likely to happen as not to happen is assigned the probability $\frac{1}{2}$; for example, if a coin is as likely to fall heads as tails (and will never land on edge), then the probability of heads is $\frac{1}{2}$ and the probability of tails is $\frac{1}{2}$.

Of course, most of the events we study are more interesting and more involved than these. How are their probabilities determined? There are two ways: by observation, and mathematically.

Probabilities that are determined by observation are of this sort:

> The probability that a child born today will live through its first year.
>
> The probability that a tossed coin will land heads.

The first of these is judged empirically, from birth-and-death records of recent years. The second is obtained by considering the symmetry of the coin and checking with some experimental tosses.

Probabilities that can be determined mathematically are of this sort:

> The probability that at least 85 of any 100 children born today will live through their first year.
>
> The probability that of 100 tossed coins, between 40 and 60 will land heads.

These complex events are made up of combinations of the simple events of a single child or coin, whose probabilities are assumed known (from observation). The probabilities of the complex events can be computed from those of the simple events according to the mathematical laws of probability.

We hope to give the child an appreciation of both these methods. We have him make judgments of empirical probabilities by performing actual experiments, recording their outcomes, and analyzing the results. We also guide him, informally, to some of the ideas underlying the mathematical formulas.

What is important is the spirit. We wish to develop in the child a feeling for the subject - an educated intuition. If he learns to make qualitative comparisons based on understanding, we have succeeded. If the best he can ever do is perform quantitative computations based on memorized rules, we have failed.

3. Comparisons of likelihood.

All children have had experience in making informal comparisons of likelihood, and this is a good topic to begin with.

> Example 1. Which is more likely on the 4th of July (here in our town) - rain, or snow?

This simple question leads to three interesting points for discussion.

(1) The comparison itself should be obvious to the children. Rain is more likely than snow because the 4th of July is usually too hot for snow.

(2) Notice that we can compare the likelihoods of two events even though we do not take into account all possible alternatives. There might be neither rain nor snow.

(3) The likelihoods being compared might both be very small. In many localities the chances for either rain or snow are virtually nil. But we can still compare the one with the other.

> Example 2. Billy is a very good student. Which is he more likely to do in tomorrow's test - pass, or fail?

Here again the comparison is evident: good students tend to pass, not fail. In this example there are no additional alternatives.

> Example 3. A new boy has joined the class. Mary is to guess his birthday. Alice is to guess how many brothers and sisters he has. Who is more likely to guess right?

This time we have no past experience to draw from. The answer is still easy, but we use a different method. Clearly, Mary has a wider selection to choose from and hence has more opportunities for guessing wrong. (We don't even have to count to see that.) Therefore, Alice is the one more likely to guess right. In this example, the most likely result is that they will both be wrong.

Events being compared may be related to each other in some way. In Example 1, rain and snow are related, in the sense that contributing causes favoring one also favor the other. Another relation between them is that of exclusion: they can't both happen at the same time (provided we define things right). Finally, it may turn out that neither one of them takes place. In Example 2, pass and fail are strongly related: one or the other must occur, but not both. In Example 3, the two guesses are not related at all. Mary's guess is of no help to Alice, nor Alice's to Mary. One guess could be right while the other is wrong; both girls might guess right; both might be wrong.

In each of these examples, it can happen that the less likely of the two events occurs while the more likely one does not. It can snow. Billy

might fail. Mary may get the birthday right even though Alice is wrong about the family.

> Example 4. When Mary guesses the new boy's birthday, which (in her guess) is the more likely to be right - the entire birthday, or the month alone?

This example is fundamentally different from the others and illustrates an elementary but important principle. If the boy was born on January 14th, then he was born in January. It is impossible for the less likely event (hitting the birthday) to occur while the more likely (hitting the month) does not. In this case, the comparison in likelihood is derived not from past experience or numerical considerations, but from logical necessity.

4. Estimating probabilities by experiment.

The probability of an event is, roughly speaking, the fraction of the time we expect it to happen. An estimate, then, is the fraction of the time it did happen (under similar conditions). Records of births over a period of years show that the fraction of boys is .5 (to within two one-hundredths); we conclude that the probability is .5 that the Jones's next child will be a boy. A major league player has a batting average of .303; we conclude that the probability of a hit his next time at bat is .303.

Children will be less interested in tables of actual statistics than in performing experiments for themselves. An excellent project is to estimate a probability about which they have no preconceived notion; for example, the probability that a thumbtack, when jiggled and thrown onto the table, will come to rest on its head. (If tacks are too hazardous, substitute rivets, bottle tops, etc.)

Our estimate of the probability is the number of heads divided by the number of throws. How many throws should be made? Questions like this can be subjected to delicate analysis, as part of the theory of statistics. Our own discussion will be much more elementary.

Before plunging into the experiment with the tacks, let us gain some insight by experimenting with the humble coin. We assume that the coin is "honest" - that is, the probability of heads is $\frac{1}{2}$.

By the way, a fair toss or spin requires some skill beyond many children. A little practice may be worth while even for the teacher. If the experiment calls for a large number of throws, and if you are not keeping track of the order of the results but only the totals, throw the coins 10 at a time. The probabilities are the same whether the throws are successive or simultaneous. Not only do you speed things up, but the whole affair is easier to control: mix by shaking in your cupped hands, and release gently onto the table. Cover the table with paper to cut down on noise. (With some modifications, you can keep track of order as well. For example, record the results of 4 coins in the order penny, nickel, dime, quarter.)

Back to our coin. Let us toss it 10 times. (Or toss 10 coins at once.) Since the probability of heads is $\frac{1}{2}$, we expect heads $\frac{1}{2}$ the time in many tosses. So we expect heads 5 times out of 10 in the long run. But of course we cannot count on getting exactly 5 heads in our particular throw of 10.

Actually, the most likely result in 10 throws is indeed to get exactly 5 heads. But this result is not likely! It will happen only about 1 time in 4. (The methods for calculating these probabilities will be gone into later on.) This fact will come as a surprise to most children. But confronting them with the theoretical quotation is not enough. For the fact to soak in, they should perform (or observe) some actual experiments.

Let us assume that in our throw of 10 we counted 7 heads. This is a perfectly reasonable result. (1 time in 3 we will get at least 7 heads or at least 7 tails.) If we stop here, our estimate of the probability of heads is .7 - pretty far off.

But suppose we throw 100 times. To obtain as many as 70 heads would be highly unusual. A reasonable result would be something between 43 and 57, say. Our estimate would now be sensibly near the true value.

Suppose we throw 1000 times. To obtain as many as 700 heads would be a miracle. Even a result like 430 or 570 is highly improbable. A reasonable result would be something between 475 and 525, say. Our estimate would now be quite close.

And now to the tacks. We throw 10 of them and count 7 on their heads, 3 on their sides. Is the probability of heads .7 ? That would be a rash conclusion. As we have just seen, we might well get the same count when

the probability is actually .5 . So we make another throw of 10. And another. We don't know just when it is safe to stop. In fact, there is no way of knowing when to stop. But we notice after a while - perhaps only after two or three hundred trials - that the results start to settle down. We watch the small random fluctuations get swallowed up in the overwhelming stability of the mass. Then we announce our estimate in a firm voice.

5. Equally likely outcomes.

Let us throw a die which, we are assured, is honest - that is, any one of the six faces is as likely to show as any other. Then each face will appear, in the long run, $\frac{1}{6}$ of the time. Hence the probability of any particular face, on a given throw, is $\frac{1}{6}$. The probability of a 4, for instance, is $\frac{1}{6}$.

This example illustrates a general principle. Suppose that an experiment can result in any one of a number of outcomes, all equally likely. Call the total number of possible outcomes n. Then the probability of any particular outcome is $\frac{1}{n}$.

In the example, the experiment is the throw of the die; the possible outcomes are the faces: 1, 2, 3, 4, 5, 6, all of which are equally likely; their number, n, is 6; and the probability of any particular outcome, $\frac{1}{n}$, is $\frac{1}{6}$.

In throwing the die, what is the probability of a number greater than 4 ? In other words, what is the probability of throwing either 5 or 6 ? Each of these appears (in the long run) $\frac{1}{6}$ of the time. Therefore one or the other appears $\frac{2}{6}$ of the time - that is, $\frac{1}{3}$ of the time. So the probability of a number greater than 4 is $\frac{1}{3}$.

The principle here is an extension of the one above. Suppose, again, that an experiment can result in any one of a number of outcomes, all equally likely; call the total number of possible outcomes n. Certain of the outcomes are "favorable" to the event we are interested in, while the rest are not; that is, the event takes place if one of the favorable outcomes occurs, but not otherwise. Call the number of favorable outcomes s. Then the probability of the event is $\frac{s}{n}$. (The earlier principle handles only the special case in which $s = 1$.)

In words: the probability of the event is equal to the number of favorable outcomes divided by the total number of possible outcomes.

In the example, the experiment is the throw of the die; the possible outcomes are 1, 2, 3, 4, 5, 6, <u>all equally likely</u>; $n = 6$; the event of interest is a number greater than 4; the favorable outcomes are 5 and 6; there are two favorable outcomes, so s is 2; and the probability of the event $\frac{s}{n}$ is $\frac{2}{6}$, that is, $\frac{1}{3}$.

It is instructive to test these conclusions by actual experiments. Make a large number of throws and check that a number greater than 4 comes up about $\frac{1}{3}$ of the time.

We have described two methods for obtaining probabilities: empirically, and by counting equally likely outcomes. In applications, it is sometimes hard to tell whether various outcomes really are equally likely. This problem, discussed in the next section, will lead to the question of how known probabilities are combined to obtain new ones.

6. <u>Probabilities of combinations</u>.

Suppose that we are interested in the number of heads showing on a throw of two coins, say a penny and a dime. There may be none at all, or just 1, or 2. Are these three outcomes equally likely? If you are in any doubt at all about the answer, make 50 throws and see what happens.

It is a common pitfall simply to assume that outcomes presented in a natural way are equally likely. The safeguard is to consider them carefully and supply a reason in support of the assumption.

The danger in our problem is to be swindled by one's own language. Let us replace the dime by a checker. Even better - keep the dime, but just <u>say</u> "star" for its heads and "crown" for its tails. On a throw of the two coins there are 4 possible outcomes: heads-star, heads-crown, tails-star, tails-crown; <u>and there is not the least doubt that these 4 outcomes are all equally likely</u>. Their original names are heads-heads, heads-tails, tails-heads, tails-tails (the first word in each case referring to the penny, the second to the dime). It is now clear that the probability of 2 heads is $\frac{1}{4}$, the probability of just 1 head is $\frac{1}{2}$, and the probability of 0 heads is $\frac{1}{4}$.

What is the situation with 3 coins - say a penny, a nickel, and a dime? For instance, what is the probability that exactly 2 fall heads? Each coin will fall in either of 2 ways, independently of how the other two fall. Altogether then, the 3 coins can fall in $2 \times 2 \times 2 = 8$ possible ways, <u>all equally likely</u>. They are:

HHH; HHT, HTH, THH; HTT, THT, TTH; TTT.

Here H denotes heads and T tails; the first letter in each triple refers to the penny, the second letter to the nickel, and the third to the dime. In this list, we have grouped the results as follows: first, the outcome consisting of 3 heads; next, the outcomes with just 2 heads; then those with just 1 head; then the one with 0 heads. There are 3 with just 2 heads. Therefore the probability of getting exactly 2 heads is $\frac{3}{8}$: the number of favorable outcomes divided by the total number of outcomes.

This is a good time to pause for experiment. Take 3 pennies, throw them 20 or 30 times, and see what happens.

Looking back, it should be clear why the number of favorable outcomes (those yielding exactly 2 heads) is 3. That is precisely the number of ways we can select 2 coins (for heads) from the 3 coins: as penny-and-nickel (HHT), penny-and-dime (HTH), nickel-and-dime (THH). Likewise, just 3 of the possible outcomes yield exactly 1 head: penny (HTT), nickel (THT), dime (TTH). Finally, there is just 1 way (HHH) to pick 3 heads, and just 1 way (TTT) to pick 0 heads.

The reasoning in the general case is similar. Suppose we toss n coins. Each one will fall in either of 2 ways, independently of how the others fall. Altogether, then, the n coins can fall in

$$\underbrace{2 \times 2 \times \ldots \times 2}_{n \text{ factors}} = 2^n$$

possible ways, <u>all equally likely</u>. Consider any number s between 0 and n, inclusive. How many of the 2^n outcomes yield exactly s heads? Answer: the number of ways of selecting s coins (for heads) from n coins. The symbol for this number is $\binom{n}{s}$:

$$\binom{n}{s} = \text{number of ways of selecting } s \text{ objects from } n \text{ objects.}$$

The probability of obtaining exactly s heads on the throw of n coins is, then $\binom{n}{s} / 2^n$: the number of favorable outcomes divided by the total number of outcomes.

The symbol $\binom{n}{s}$ is read "n above s" (not "n over s"). It is not a fraction, even though it looks something like one. It is a special symbol; the one most popular among mathematicians for the number of ways of selecting s objects from n objects. In place of $\binom{n}{s}$, some writers use $C(n,s)$ or $C_{n,s}$ ("the number of Combinations of $\underline{n}$ things taken $\underline{s}$ at a time"). None of the symbols is used in this teaching material.

7. Computing the number of combinations.

We have to know how to compute $\binom{n}{s}$ for various values of n and s. When n and s are small, the computations are easy. For instance, the example of the 3 coins shows that $\binom{3}{2} = \binom{3}{1} = 3$, and that $\binom{3}{3} = \binom{3}{0} = 1$.

It is clear that $\binom{1}{1} = 1$, $\binom{2}{2} = 1$, and so on: $\binom{n}{n} = 1$ whatever the value of n; for, there is just 1 way of selecting n objects from n objects (namely, choose them all). It is also easy to see that $\binom{1}{0} = 1$, $\binom{2}{0} = 1$, and so on: $\binom{n}{0} = 1$ for every value of n; for, there is just 1 way of selecting 0 objects from n objects (namely, omit them all).

Next, we see that $\binom{1}{1} = 1$, $\binom{2}{1} = 2$, $\binom{3}{1} = 3$, and so on: $\binom{n}{1} = n$ whatever the value of n; for, there are just n ways of selecting a single object from n objects (namely, pick any one of them). There are also just n ways of picking $n-1$ objects from n objects (namely, omit any one of them); therefore $\binom{1}{0} = 1$, $\binom{2}{1} = 2$, $\binom{3}{2} = 3$, and, in general, $\binom{n}{n-1} = n$.

For the next observation, consider as an illustration the case $n = 10$, $s = 8$. (Suppose, for example, that we are interested in the number of ways 10 coins can show exactly 8 heads.) For each selection of 8 coins for heads, there remain 2 showing tails. A different choice of 8 heads yields a different remaining set of 2 tails. If we run through all possible selections of 8 coins, then (from those left over in each case) we simultaneously run through all possible selections of 2 coins. It follows that there are exactly as many ways of choosing 8 coins from 10 as of choosing 2 coins from 10. Therefore $\binom{10}{8} = \binom{10}{2}$. Similarly, $\binom{10}{7} = \binom{10}{3}$, and $\binom{10}{6} = \binom{10}{4}$. The reasoning here is general and yields the general formula $\binom{n}{n-s} = \binom{n}{s}$.

The values of $\binom{n}{s}$ increase surprisingly rapidly. For example, $\binom{52}{5} = 2,598,960$, and $\binom{52}{13} = 635,013,559,600$. (These are the number of possible 5-card hands and the number of possible bridge hands, respectively.)

Small values of $\binom{n}{s}$ are easy to calculate by means of the so-called "Pascal triangle," named for the mathematician Blaise Pascal (1623 - 1662). The triangle starts out

$$\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

representing the values of

$$\begin{array}{ccccccccccc}
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

and can be continued indefinitely.

The arrows shown illustrate the method by which new values are generated. Each entry (if not the first or last in its row) is equal to the sum of the two nearest it in the line above. In the illustration, $6 + 4 = 10$, representing $\binom{4}{2} + \binom{4}{3} = \binom{5}{3}$.

To see why this works, think of a man with his 4 sons on a camping trip. From the 5, a team of 3 are to be chosen to gather kindling. The number of possible teams is, then, $\binom{5}{3}$. Now, either Dad is on the team, or he isn't. If Dad is on the team, then his 2 partners are chosen from among the 4 boys; there are $\binom{4}{2}$ ways of choosing them. If Dad is not on the team, then the whole team of 3 is chosen from among the 4 boys; there are $\binom{4}{3}$ ways of choosing them. Altogether, then, there are $\binom{4}{2} + \binom{4}{3}$ ways of choosing the team. Consequently, $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$.

A formula for computing $\binom{n}{s}$ is given at the end of the next section.

One way to check one's arithmetic, when calculating from the triangle, is to see that the sum of the entries in the n^{th} row is equal to 2^n. For example, the sum in the 5^{th} row is

$$1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5 .$$

This must be so because, as we saw on page 13, there are 2^n possible outcomes altogether in a throw of n coins; and in adding across the n^{th} row, we are counting up these same outcomes (the number with 0 heads, + the number with 1 head, and so on).

By the way, the entries in the Pascal triangle arise in algebra when one multiplies out an expression like $(H + T)^5$. The result is:

$$(H + T)^5 = H^5 + 5H^4T + 10H^3T^2 + 10H^2T^3 + 5HT^4 + T^5 .$$

The coefficients - 1, 5, 10, 10, 5, 1 - are precisely the entries, in order, in the 5^{th} row of the triangle. Moreover, the exponents show the numbers of occurrences. For example, the term $10H^3T^2$ gives us the information that there are 10 ways of getting 3 heads and 2 tails (in a throw of 5 coins).

8. Permutations.

A <u>permutation</u> is an ordered arrangement. One often finds it necessary to know how many permutations are possible of a given number of objects. For 2 objects - say the digits 1, 2 - there are just 2 permutations: 12, and 21. For 3, there are 6 permutations:

123, 132; 213, 231; 312, 321.

For example, there are 6 ways that Jim, Dick, and Sue can be assigned to clear the table, wash the dishes, and dry the dishes (one to a task).

The number of permutations of 3 objects is 6 because 6 is equal to $3 \times 2 \times 1$. (We could write 3×2, but $3 \times 2 \times 1$ looks better.) The reasoning goes like this. There are 3 choices for the first position. With each such choice, there are 2 ways of permuting the 2 remaining objects (in the 2 remaining positions). Hence there are 3×2 permutations of all 3.

The number of permutations of 4 objects is $4 \times 3 \times 2 \times 1 = 24$. For, there are 4 choices for first position. With each such choice, there are $3 \times 2 \times 1 = 6$ ways of permuting the 3 remaining objects (in the 3 remaining positions). Hence there are $4 \times 3 \times 2 \times 1$ permutations of all 4.

Similarly, the number of permutations of n objects is equal to the product of all the numbers from n down to 1. This number is ~~known~~ as "n factorial," and is denoted by $n!$ For example, $1! = 1$ and $2! = 2 \times 1 = 2$. Also, as we have seen,

$$3! = 3 \times 2 \times 1 = 6$$

and

$$4! = 4 \times 3 \times 2 \times 1 = 24.$$

These expressions, as well as the reasoning that led to them, suggest how to compute each new factorial from the preceding one. In computing $4!$, for example, we don't have to multiply out $4 \times 3 \times 2 \times 1$ if we already know that $3! = 3 \times 2 \times 1 = 6$; instead, we say, more simply,

$$4! = 4 \times 3! = 4 \times 6 = 24.$$

The saving becomes clearer as soon as one deals with larger numbers. Here are the next few factorials:

$$\begin{aligned}
5! &= 5 \times 4! = 5 \times 24 = 120; \\
6! &= 6 \times 5! = 6 \times 120 = 720; \\
7! &= 7 \times 6! = 7 \times 720 = 5{,}040; \\
8! &= 8 \times 7! = 8 \times 5{,}040 = 40{,}320; \\
9! &= 9 \times 8! = 9 \times 40{,}320 = 362{,}880; \\
10! &= 10 \times 9! = 10 \times 362{,}880 = 3{,}628{,}800.
\end{aligned}$$

As is seen, the numbers $n!$ increase more rapidly than most people might guess. Suppose that 10 children come up to sharpen their pencils. There are more than three-and-a-half million ways of deciding which child will be first, which second, and so on, down to which will be last! (No wonder we use an exclamation point!)

Factorials can be used to compute the numbers $\binom{n}{s}$. The formula is:

$$\binom{n}{s} = \frac{n!}{s!(n-s)!}$$

For example,

$$\binom{6}{2} = \frac{6!}{2!\,4!} \qquad \binom{6}{3} = \frac{6!}{3!\,3!}$$

and so on. In the actual calculations, there is always some cancelling that can be done first to simplify the work:

$$\binom{6}{2} = \frac{6!}{2!\,4!} = \frac{6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2}}{2 \times \cancel{4} \times \cancel{3} \times \cancel{2}} = \frac{6 \times 5}{2} = 15 .$$

There is one detail that sometimes looks strange at first. So far, we have not said what $0!$ is. The definition given above for $1!$, $2!$, and the others does not work for $0!$; therefore $0!$ needs a separate definition. The definition given is: $0! = 1$. The reason for this is to make the formula for $\binom{n}{s}$ work in every case. For example:

$$\binom{5}{0} = \frac{\cancel{5!}}{0!\,\cancel{5!}} = \frac{1}{0!} = 1 .$$

9. <u>Yes or no</u>.

In these concluding sections we describe some of the basic rules of probability.

We shall use the letter P as an abbreviation for "the probability of." In place of "the probability of heads," for instance, we will write

$$P(\text{heads}) .$$

(The parentheses are there to help the eye.)

Suppose we wonder whether Billy will pass or fail tomorrow's test. He may be more likely to do one than the other, depending on circumstances. But in any case, two things are clear: he can't do both; and he must do one or the other. Then the following is true:

$$P(\text{pass}) + P(\text{fail}) = 1 .$$

The rule is this. An event either happens or does not happen. We must have one or the other, and cannot have both. Then

$$P(\text{it happens}) + P(\text{it does not happen}) = 1 ;$$

more briefly,

$$P(\text{yes}) + P(\text{no}) = 1 .$$

In a typical problem, we know one of the values on the left (the first, say) and solve for the other:

$$P(\text{no}) = 1 - P(\text{yes}) .$$

For example, if the probability that Billy passes is .6, then the probability that he fails is .4 . If the probability that Dad takes Jim to the circus is .3, then the probability that he doesn't is .7 .

$2 \quad 2!\,4! \qquad 2 \times 4 \times 3 \times 2 \qquad 2$

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LESSON 1

Certainty and Uncertainty

Objective: To call attention to the fact that some events are predictable whereas others are uncertain.

Vocabulary: While many words might be included in a vocabulary list, the fact that a child uses "sure" instead of "certain", or "not very likely" instead of "improbable" should not be alarming. It is important that he have the ideas from the lesson, but not that he say particular words. Therefore, you should feel free to use expressions which are meaningful to the class. Introduce words such as forecast, predict, and uncertain, but do not insist on the children's using them.

Materials: Blocks (some all one color, others of the same size and shape but of different colors); non-transparent cloth bags; checkers or counting disks marked with a crown on one side and a star on the other.

Suggested Procedure:

Time devoted to this lesson should be determined by the interest and understanding of the class. For some children, the idea of the uncertainty of some events may be obvious, but for others it may not. Activities suggested may be used on different days.

Lead a discussion of the certainty of some events in everyday life as opposed to the uncertainty of others. You may want to introduce the idea by questions like the following:

Can you tell something that's going to happen tomorrow?

Do you know what a fortune teller is?

Do you think it's going to rain?

What kind of car will come around the corner next?

Will we come to school on Sunday?

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other.

How many of you had more C's than S's ?

How many had more S's than C's ?

Who had the largest number of C's ?

Who had the largest number of S's ?

Pupil page 2: This page repeats the first activity and should go more smoothly. Compare the results this time with those the pupils obtained on the first page. Ask questions as before to develop the idea of certainty and uncertainty, and discuss pupils' answers to the eight questions.

Pupil page 3: By the time the pupil completes this page and discusses his answers to the questions, he should understand that the checker is "fair", that is, that one side is just as likely to come up as the other. He also will know that he can't be certain which side will be up on the next shake or spin. You may want the pupils to add to show the total C's and S's they obtained on the three experiments and to find the total number of times they spun the checker.

Shake your checker in a cup or spin it and let it stop. If the ☆ shows, put an S in one of the boxes under ☆ . If the ♔ shows, put a C in a box under the ♔ . Stop when <u>one set</u> of boxes is filled.

☆

♔

1. How many C's did you make? ________
2. How many S's did you make? ________
3. How many times did you spin or toss the checker? ________
4. How many more marks are there in one set of boxes than in the other? ________
5. Which do you think another toss or spin will show, a ☆ or a ♔ ? ________
6. Toss or spin again. Did you guess right? ________

Now let's try again. Shake your checker in a cup or spin it and let it stop. If the ☆ shows, put an S in one of the boxes under ☆. If the  shows, put a C in a box under the . Stop when one set of boxes is filled.

☆

1. Did you get the same results as last time? ______
2. How many times did you spin or toss the checker? ______
3. How many S's did you make? ______
4. How many C's did you make? ______
5. How many more marks are there in one set of boxes than in the other? ______
6. Which do you think another toss or spin will show, a ☆ or a  ? ______
7. Toss or spin again. Did you guess right? ______
8. Can you be certain which side of the checker will be up? no

Shake your checker in a cup or spin it and let it stop. Show in the boxes which side comes up. If the side with the ☆ comes up, put an S in a box. If the  comes up, put a C in a box. Stop when all boxes are filled.

1. How many ☆'s did you see? ________
2. How many 's did you see? ________
3. How many times did you spin or toss the checker? ________
4. What is the difference between the number of S's and C's ? ________
5. Guess what will come up next time you spin or toss. ________
6. Spin or toss again. Did you guess right? ________
7. Each time you do this, is it uncertain which side will come up? ________

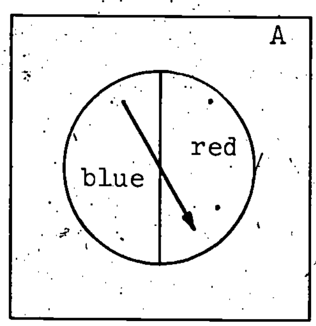
LESSON 2

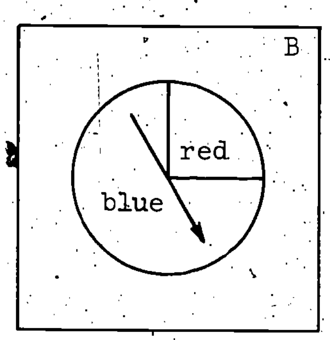
Comparing the Likelihood of Various Events

Objective: To show that some events are equally likely, and that some events are more likely than others.

Vocabulary: Equally likely, more likely, less likely, probably, chance.

Materials: Colored blocks and bags, as for Lesson 1; large spinners for use at the front of the room with dials of (A) half red, half blue, and (B) one-fourth red, three-fourths blue (see the last section of this commentary); small spinners and markers such as a bean, piece of corn, or bit of colored paper for use by the children.





Suggested Procedure:

Discuss the fact that although some events are not absolutely certain, we say they are "very likely" or "almost sure" to happen. In a given season, the weather is very likely to be warm (or cold). It is not certain that there will be milk for lunch (because the truck might break down), but it's almost certain. You cannot be sure you will see at least one car on the way home from school, but you'd be very much surprised if you didn't. When we say something is probable, or that it will probably happen, we think it is more likely to happen than not. There are some things that are not very likely to happen, but they are not impossible. Other things are impossible (the cow jumps over the moon, for instance). Use examples from children's daily life of more likely and less likely events and of things that are impossible.

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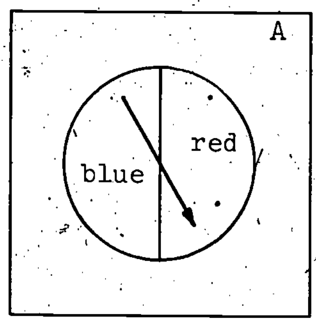

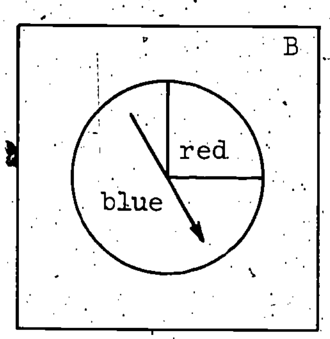

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the spinner and show the pupils the result. Large spinners must also be carefully used and checked to be sure that the pointer is not influenced by the position of the spinner.

Use Spinner A. Ask children where they have seen spinners before and why they were used. Develop the idea that spinners select "by chance" because the person who spins cannot know in advance just where the pointer will stop. Ask children whether they can tell which color the pointer will stop on. (No.)

Is it more likely to stop on red than on blue? (No.)

Is it less likely to stop on red than on blue? (No.)

Are red and blue equally likely? (Yes.)

Have each child spin it once and keep a record of the results on the chalkboard. Discuss these results, bringing out the notion that red and blue are equally likely.

red	blue
//	///

Use Spinner B and repeat. Discuss these results. Ask if blue and red are now equally likely; if red is less likely than blue; if blue is more likely than red. Let children tell in their own words why they think blue is more likely than red.

Suggest playing the same game with blocks as in the preceding lesson, but this time explain that you will put two green blocks and one yellow block into the bag. There will probably be a protest from the children whose teams would get a point for a yellow block. If not, let one pair of teams play briefly; record and discuss the scores.

Pupil pages 4-8: On page 4 children can pass the spinner to another pair as they finish. Keep a record on the board of the number of winners and discuss with the class the fact that the players are equally likely to win. If you think it is desirable, the game can be played again with the children choosing a different color this time.

Most players will probably not reach the top or the bottom on page 5 because the spinner is "unbiased". Because this is the first time pupils

keep track of the spins by marking an $\times$ in the circle, they may need to be reminded to mark one circle each time they spin. In discussing the four questions, children may be surprised to see how few in their room actually reached "stop" and how many times the marker moved first in one direction and then in the other.

Page 6 is similar to page 5. Record on the board the number of pupils who ended on red, on blue, at the "Circus", and at "Clean Your Room". Let them discuss these results.

Pages 7 and 8 use the one-fourth red, three-fourths blue spinner. Discuss the answers to the questions on these pages. Ask children why so many more of them reached the goal here than on pages 5 and 6.

<u>Pupil pages 9-10</u>: These two pages will help you evaluate the children's understanding of the first two lessons. For some of the items, such as 2, 3, and 5 on page 9, there are many answers but these indicate that the child understands.

A discussion should follow the completion of each of these pages.

<u>Optional Further Activity - An Experiment</u>

Some children might profit from this experiment. How many times, on the <u>average</u>, do you think you have to spin a one-half red, one-half blue spinner before it points to red?

Spin the one-half red, one-half blue spinner and count the spins before you get red. If you get red on the first spin, write 1 . If you do not get red until the fourth spin, write 4 . Then start over. Keep a record in columns of 10 reds each. When you have recorded the number of spins for ten reds, add the number of spins needed, and divide this sum by 10 to find the <u>average</u> number of spins needed.

Does the average change much from one column to another? Add the sums for five columns (50 reds) and divide by 50 . Is this average greater or less than 2 ? How much greater or less?

Children could arrange their record in this form:

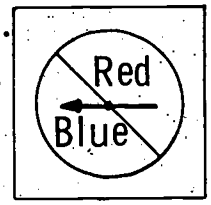
Number of Spins to Get a Red

10th red					
9th red					
8th red					
7th red					
6th red					
5th red					
4th red					
3rd red					
2nd red					
1st red					
SUM					
SUM ÷ 10					

From this experiment, more mature youngsters could gain some feeling for the stability in large numbers and the randomness of the smaller samples.

A Race

Use the spinner



Keep it flat on the desk.

You and another pupil race.

First color the race track red or blue as shown.

One pupil chooses the red track, the other the blue track.

Each pupil put a marker on <u>start</u>.

If the spinner stops on red, the pupil with the red track moves one space.

If the spinner stops on blue, the pupil with the blue track moves one space.

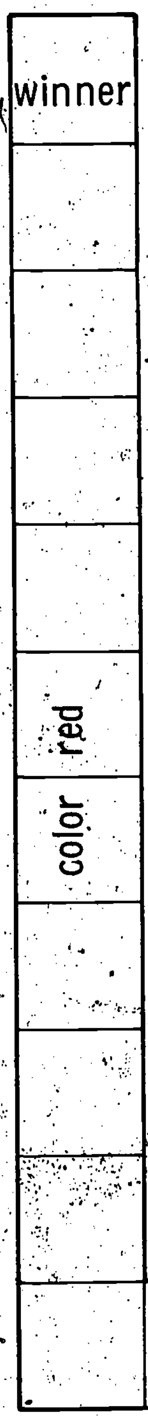
Take turns using the spinner.

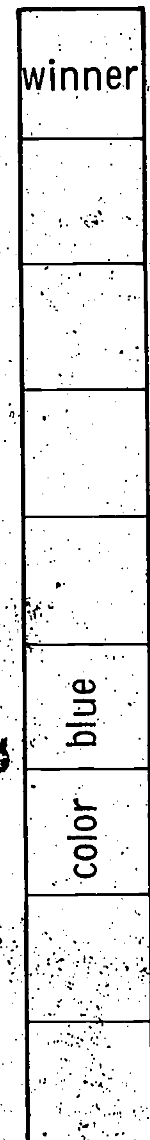
Spin until one pupil wins.

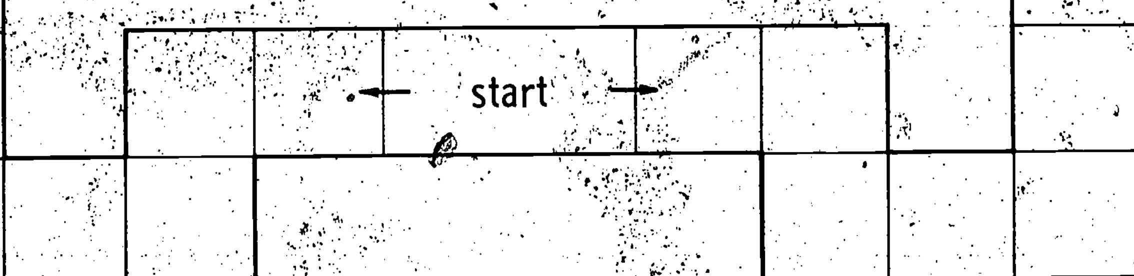
Now start the race - spin.

Repeat if your teacher directs.

Keep track of the winners.

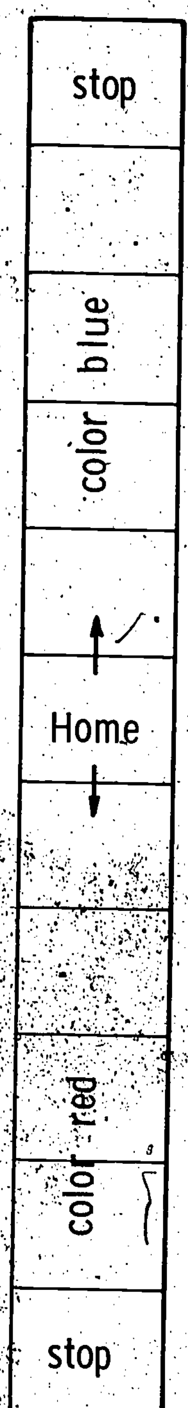




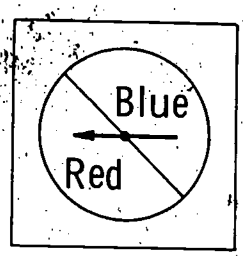


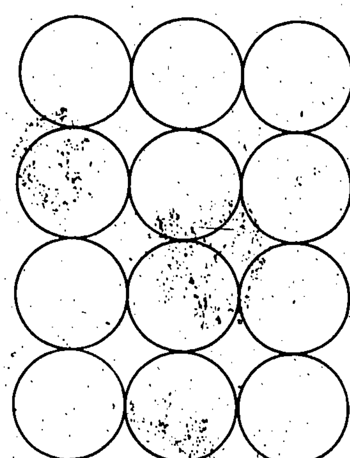
Take a Walk

Color the sidewalk as shown.
Do not color the space marked "Home".



Use the spinner:



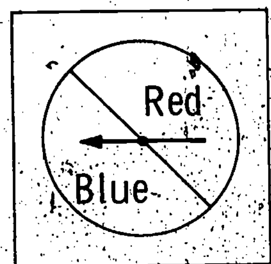


Put your marker on Home. If the spinner stops on red, move <u>down</u> one space. If it stops on blue, move <u>up</u> one space. Put an X in one of the circles each time you spin. When you fill the circles, put an X in the space where your marker is. If you get to the top or the bottom, do not spin again. Put an X in the space where your marker is.

1. How far from Home is your marker? ________
2. Where is it - on red, on blue, or back Home? ________
3. How many pupils in your room are on blue? ________
4. According to the rules, how far from Home could you get? 5 spaces

A Choice

Use the spinner.



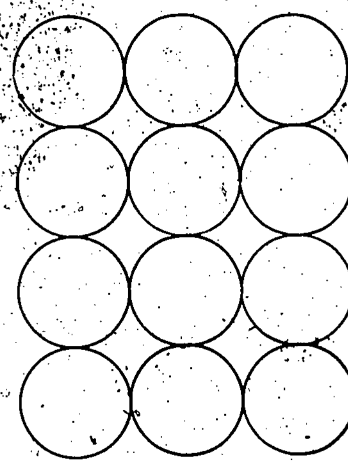
Your mother gives you a choice of going to the circus or cleaning your room.

Color the choice as shown.

Put your marker on start. If the spinner stops on red, move one space toward "Clean your room".

If the spinner stops on blue, move one space toward "Circus".

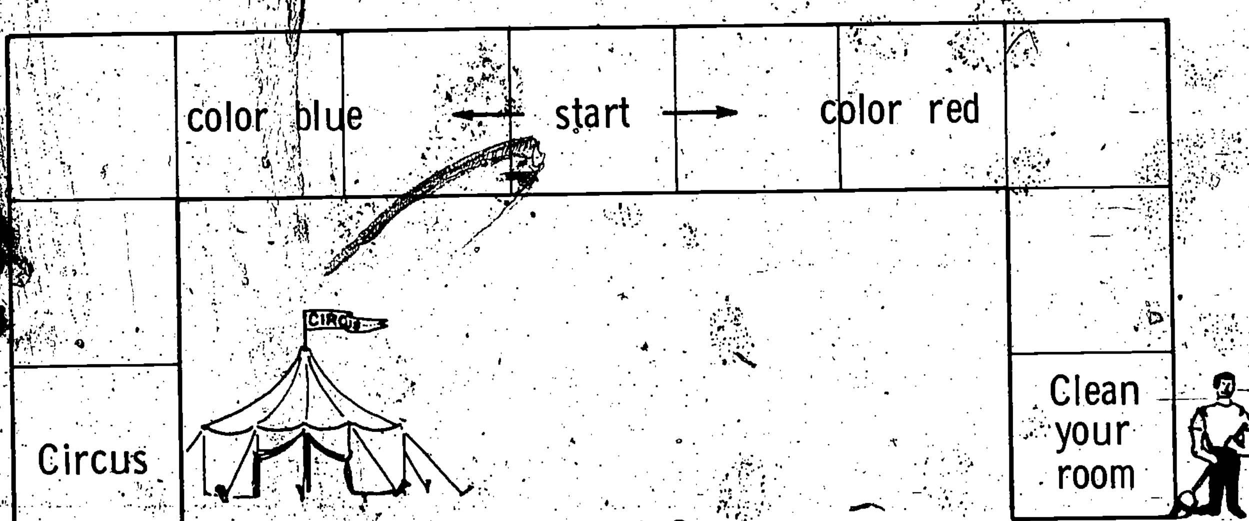
Put an X in one of the circles here each time you spin.



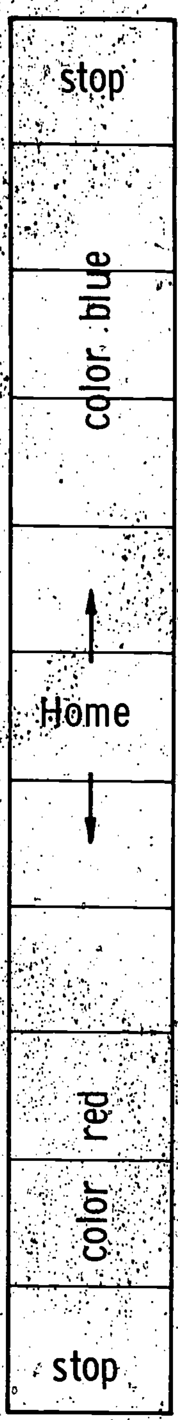
If you get to "Circus" or "Clean your room", stop.

If you fill in all of the circles, stop and mark with an X where your marker is.

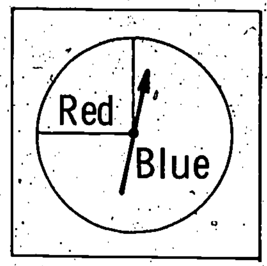
Spin.



Another Walk



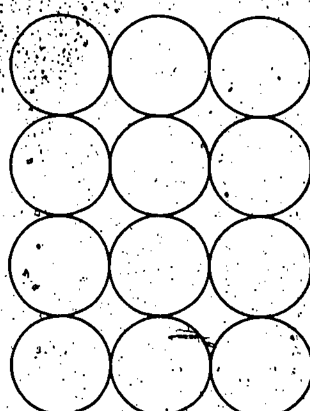
Use the spinner:



Color the sidewalk as shown. Do not color the space marked "Home".

Put your marker on Home. Spin. If the spinner stops on red, move down one space. If it stops on blue, move up one space.

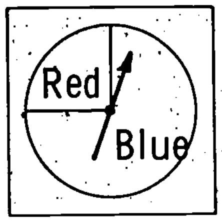
Put an X in one of the circles ~~each~~ time you spin. When you fill the circles, put an X in the space where your marker is. If you get to the top or the bottom, do not spin again. Put an X in the space where your marker is.



1. Where did you end - on red, on blue, or back Home? ____________
2. How many pupils in your room did not get to either end? ____________
3. How many pupils stopped on red? ____________
4. Do you think more pupils should have stopped on red than on blue? no

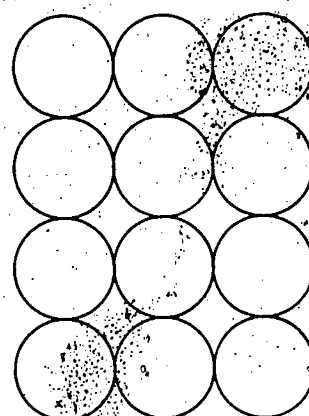
Another Choice

Use the spinner



Your mother gives you another choice. If the spinner stops on red, move toward "Clean your room". If the spinner stops on blue, move toward "Circus".

Put an X in one of the circles here each time you spin.



If you get to "Circus" or "Clean your room", stop.

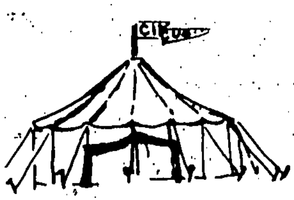
If you fill in all of the circles, stop and mark with an X where your marker is.

Spin.

Color blue ← start → Color red

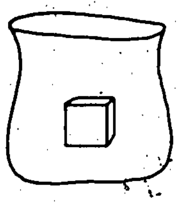
Circus

Clean your room

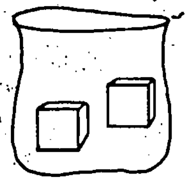


1. Did you get to the circus? _____
2. How many pupils in the room got to the circus? _____
3. How many had to "Clean your room"? _____
4. How many pupils in the room did not get to either end? _____

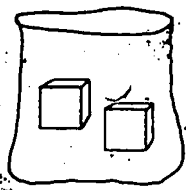
Color the blocks in the bags blue. Tell how many red blocks you would put in each bag:



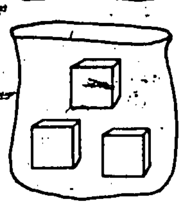
1. If I wanted to have an <u>equal</u> <u>chance</u> of drawing red or blue, I would put in ___1___ red blocks.



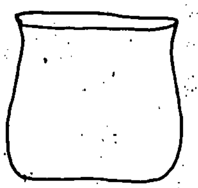
2. If I wanted blue to be <u>more</u> <u>likely</u> than red, I could put in ___1 or 0___ red blocks.



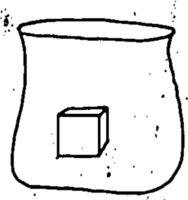
3. If I wanted blue to be <u>less</u> <u>likely</u> than red, I could put in ______ red blocks.
3 or more than 3



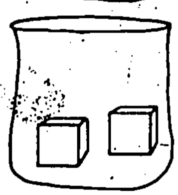
4. If I wanted an <u>equal</u> <u>chance</u> of drawing red or blue, I would put in ___3___ red blocks.



5. If I wanted to be sure I would draw a red every time, I could put in ___1 or more___ than 1

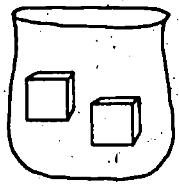


6. If I wanted to be sure of drawing a blue block, I would put in ___0___ red blocks.

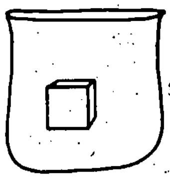


7. If I wanted to be sure of drawing a blue block, I would put in ___0___ red blocks.

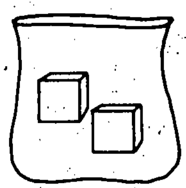
Color the blocks in the bags blue. Tell how many red blocks you would put in each bag:



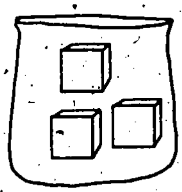
1. If I wanted to draw a red block about as often as a blue one, I would put in 2.



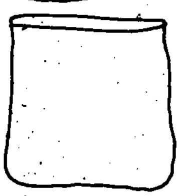
2. If I wanted to draw a red block 3 times as often as a blue one, I would put in 3.



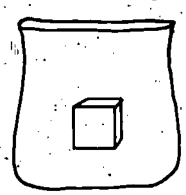
3. If I wanted to draw a red block half as often as a blue one, I would put in 1.



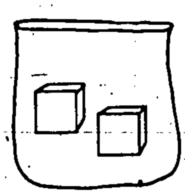
4. If I wanted to draw a blue block every time, I would put in 0.



5. If I wanted to draw a red block every time, I could put in 1 or more than 1



6. If I wanted to draw a red block twice as often as a blue one, I would put in 2.



7. If I wanted to draw a red block one third of the time, I would put in 1.

Brain Teaser: I have a bag with 3 red blocks in it. If I wanted to draw a blue block $\frac{2}{3}$ of the time, I would put in 6 blue blocks.

LESSON 3

Combining Events: This One AND That One

Objective: To show that probability may decrease when two events are both required to happen.

Vocabulary: Both, and.

Materials: Two cloth bags, red and blue blocks, several small spinners with dials half-red, half-blue.

Suggested Procedure:

(NOTE: Because the concept involved is hard for children to verbalize, an attempt is made here to provide experiences which will lead them to the desired conclusion. The games suggested will probably require more than one class period. They may also be played on such occasions as rainy recesses.)

Game 1. Divide the class into two teams or into several sets of two teams. Give each set of two teams a bag with one red block and one blue block in it. A member of a team, without looking, takes a block from the bag. If it is red he scores a point for his team, but if it is blue, no point is scored. The block is returned to the bag and the bag is shaken. A player from the other team then gets to draw. Players on the two teams continue to take turns until each team has had 20 draws. The points are recorded on the board in the box for the first game.

	Team A	Team B
First Game	𝍸 𝍸 |	𝍸 𝍸 ||
Second Game		

Game 2. Now provide two bags for each pair of teams. Each bag contains one red block and one blue block as before. Play the game again but this time two members of the same team, each using a different bag, withdraw a block. In order to score a point, both blocks chosen must be red. The points are tallied on the board in the box for the second game. The blocks are replaced in the bags from which they were withdrawn and two players from the second team get

to draw. Each team gets twenty turns as before. Compare the number of points scored in this game with the scores in the first game. The record might now look like this:

	Team A	Team B
First Game	𝍸 𝍸 |	𝍸 𝍸 ||
Second Game	𝍸 |	||||

If there were more points scored in the second game than in the first (unusual but not impossible), play the game several more times. We would expect each team to score about 10 points in the first game and about 5 points in the second game.

Discuss the two games with the class.

> Is it more likely that both members of a team will get a red block than that one will? (No.)
>
> Is it just as likely? (No.)
>
> Is it less likely that both players on a team will get a red block than that just one will? (Yes.)

<u>Game 3</u>. (<u>Pupil page 11</u> can be used now or you can use it when playing this game a second time.) Play a similar game but have one team member use the half-red, half-blue spinner and a teammate draw a block from the bag. To score, the spinner must stop on red <u>and</u> a red block must be withdrawn. Record the points as before on the board.

	Team A	Team B
First Game	𝍸 𝍸	𝍸 𝍸 |
Second Game	𝍸 |	||||
Third Game	|||	𝍸

> Did the result of spinning have any effect on the block that was chosen? (No.)
>
> Did one thing have anything to do with the other? (No.)
>
> Was Billy just as likely to get red as blue when he used the spinner? (Yes.)

Was Sam just as likely to get red as blue when he took a block out of the bag? (Yes.)

Is it just as likely to get red both times as it is to get red on one or the other? (No.)

Jack, tell us which is more likely to happen in this game -- to get red both times or to get red just once.

How are the second game and the third game alike?

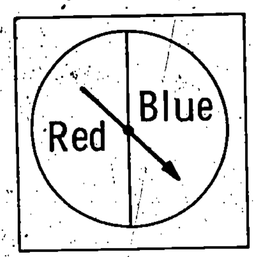
Pupil page 11: This page can be used with the third game when that game is introduced, or it can be used as a class activity with two teams when the game is played for a second time.

Pupil pages 12 and 13: A discussion of these two pages after children have completed them should help pupils understand that it is less likely that two events will both happen than that a given one of the two will happen.

Some children might enjoy making a problem for their classmates similar to number 6 on page 13.

Spin and Draw

Use the spinner:



Put a red block and a blue block in a bag. Have two teams. One player for Team A spins. Then another player for Team A withdraws a block from the bag.

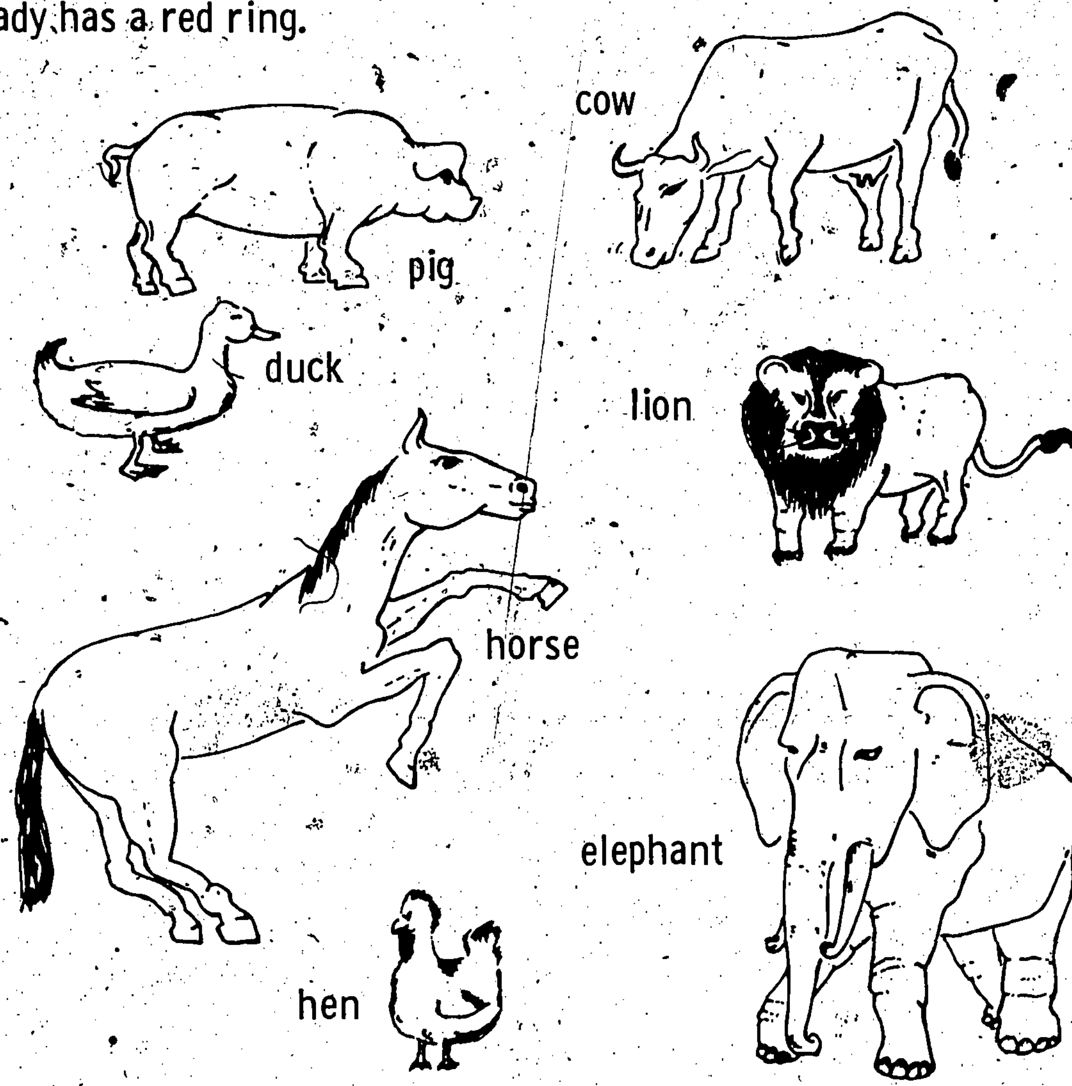
To score, the spinner must stop on red and a red block must be withdrawn. Now Team B spins and draws. Play until each team has 20 turns. Teams take turns. Keep score in the box by using tally marks.

Team A	Team B

1. What is Team A's score after 20 turns? ______
2. What is Team B's score after 20 turns? ______

3. Are you just as likely to get red as blue when you spin? yes
4. Are you just as likely to draw a red block as a blue block? yes
5. Does the spin have any effect on the block that is chosen? no
6. Which is more likely? (Draw a ring around the number.)
   (1) to get red both times.
   (2) to get red just once.

Draw a red ring around the name of every animal that has four legs. Draw a blue ring around the name of every farm animal, even if it already has a red ring.



1. I drew __5__ red rings. There are __5__ animals with four legs.
2. I drew __5__ blue rings. There are __5__ farm animals.
3. __3__ words have both a red and a blue ring around them. There are __3__ animals which are both farm animals and have four legs.
4. Are all animals which have four legs farm animals? __no__
5. Do you think there are more animals which have four legs than there are animals which both have four legs and are farm animals? __yes__

Look at these words. Draw a red ring around every word that has the letter a in it. Draw a blue ring around every word that has the letter n in it, even if it already has a red ring.

about	begin	cat	do	each
father	hand	him	always	banana
look	make	my	penny	one
ran	we	where	work	you

1. 9 words have the letter a in them.
2. 6 words have the letter n in them.
3. 3 words have both the letter a and the letter n in them.
4. Which are there more of -- words with red rings, or words with both red and blue rings? red rings.
5. Would you expect to have more words with both a and n than just with a ? no
6. In the Wicks family there are four children.
   Guess which is more likely:
   (1) Each child's name has an r in it.
   (2) Each child's name has both an r and an a in it.
   Here are the actual names of the children:
   Richard Frank Dorothy Mary
   Look carefully at each name. Did you guess correctly? ________

LESSON 4

Combining Events: This One OR That One

Objective: To show that probability may increase when either one result or another (or both) is acceptable.

Vocabulary: Either, or.

Materials: Two green blocks, one yellow block, bag, small spinners with dial half red, half blue.

Suggested Procedure:

Recall games in which children have more than one guess (Seven-Up, Dog and Bone, etc.). Ask whether these games give a child a better chance to win than if he had just one guess.

Tell the class that you have one block in the bag. Ask a child to whisper to you what color he thinks it is. Without telling him that he is right or wrong, ask someone else to whisper his guess to you. Discuss the fact that there are now two chances that the color was guessed.

> Which is more likely: that either Bob or Bill guessed right or just that Bill guessed right? (Either Bob or Bill -- at least one of the two.)

I. Divide the class into two teams. Use two green blocks and one yellow block in a bag. To score a point for his team, the player must withdraw the yellow block. The block is returned to the bag after the draw. All the children can keep a record of the points in box I on Pupil page 14.

II. Play the game again. This time allow the player to select a second time without replacing the first block, if his first choice was not the yellow block. Children can record the points in box II on page 14.

III. Play a third game. In order to score this time, the child must use the half-red, half-blue spinner first and then withdraw a block. If either the spinner stops on red or he takes the yellow block, he scores a point. If he

gets both desired results, the score is still only one point. Record the points in box III on page 14. Play the same number of turns for each game.

Discuss the results of these three games. Note, for example, that in the third game in which a point is scored <u>either</u> for red on the spinner <u>or</u> for a yellow block, the scores are higher than in the first game. The games could be played again if you think it is desirable.

In probability, "either,or" also means "or both", so one thinks, "Either this can happen or that can happen or both can happen." This is different from the way boys and girls have learned to interpret the term "either,or" which is "Either this or that but <u>not</u> both." If a parent says, "You may have milk or lemonade," a child does not understand this to mean "either, or, or both". A good illustration of the mathematical meaning in a setting familiar to boys and girls is rather difficult to give. However, discuss the fact that sometimes when you say, "Either this may happen or that may," it is possible that both things will happen. If Mother said, "Either we will have a picnic on Saturday or we'll go to the drive-in for dinner," she might also decide that you would first have a picnic lunch and then go to the drive-in in the evening.

In this discussion, mention that if you had 2 blue blocks and 1 red one in a bag, you might draw a blue one either the first time or the second time, or both.

Sometimes, however, it is impossible for both things to happen. Either it will rain tomorrow or it won't, but not both. In game II, you might pick the yellow block on the first draw, but if you did you couldn't also get it on the second draw (because it would no longer be in the bag). Of course, you might not get the yellow block at all.

<u>Pupil pages 15</u> and <u>16</u>: Children are to draw rings around the words as indicated. The questions are to help them see that if the situation is "either,or" there are more possibilities than if it is "both,and". This can be developed in a discussion of their answers.

For some children you may want to add these questions for page 16:

5. What is the sum of the answers to questions one and two? (8 + 11 = 19.)
6. From the answer to question five, subtract the answer to question three. (19 - 5 = 14.)

7. Is the answer to question six the same as the answer to one of the other questions? (Yes, question 4.) Explain why this is so. (In question 5 we have the number of words with the letter s or the letter e in them. We subtract the number of words that have both s and e in them, as these have been counted twice.)

Pupil page 17: This page will require close supervision! Children should first complete the section under Spins, recording their first spin and their second spin in the proper columns. After they have made the entries for 16 sets of two spins, they then record these under the heading Results. Each pupil must work carefully to tally these correctly in the four columns because he sometimes makes 2 tallies for a row and sometimes only one. For example, a red on the first spin and a red on the second spin would be marked under Red and Red and also under At least one Red. If the two spins had been blue on the first spin and blue on the second spin, this would be recorded by only 1 tally under No Red. This will be difficult for children to do at first.

You can check rather quickly by noting that if a mark is made in the third column, No Red, then there should be no marks in the other columns for that row. If a tally appears in the first column, Red and Red, there should also be a tally in the last column, At least one Red. If a tally appears in the second column, Exactly one Red, then a tally should also be made in the last column, At least one Red.

Discuss children's answers to the seven questions and encourage them to state their reasons.

Score Card

This page is for you to keep score of some games played by your class. Your teacher will lead the games.

I

Two green blocks and one yellow block in a bag. <u>One</u> draw.

Team A	Team B

Make tally marks

Winner is ____________

II

Two green blocks and one yellow block in a bag. <u>Two</u> draws allowed.

Team A	Team B

Make tally marks

Winner is ____________

III

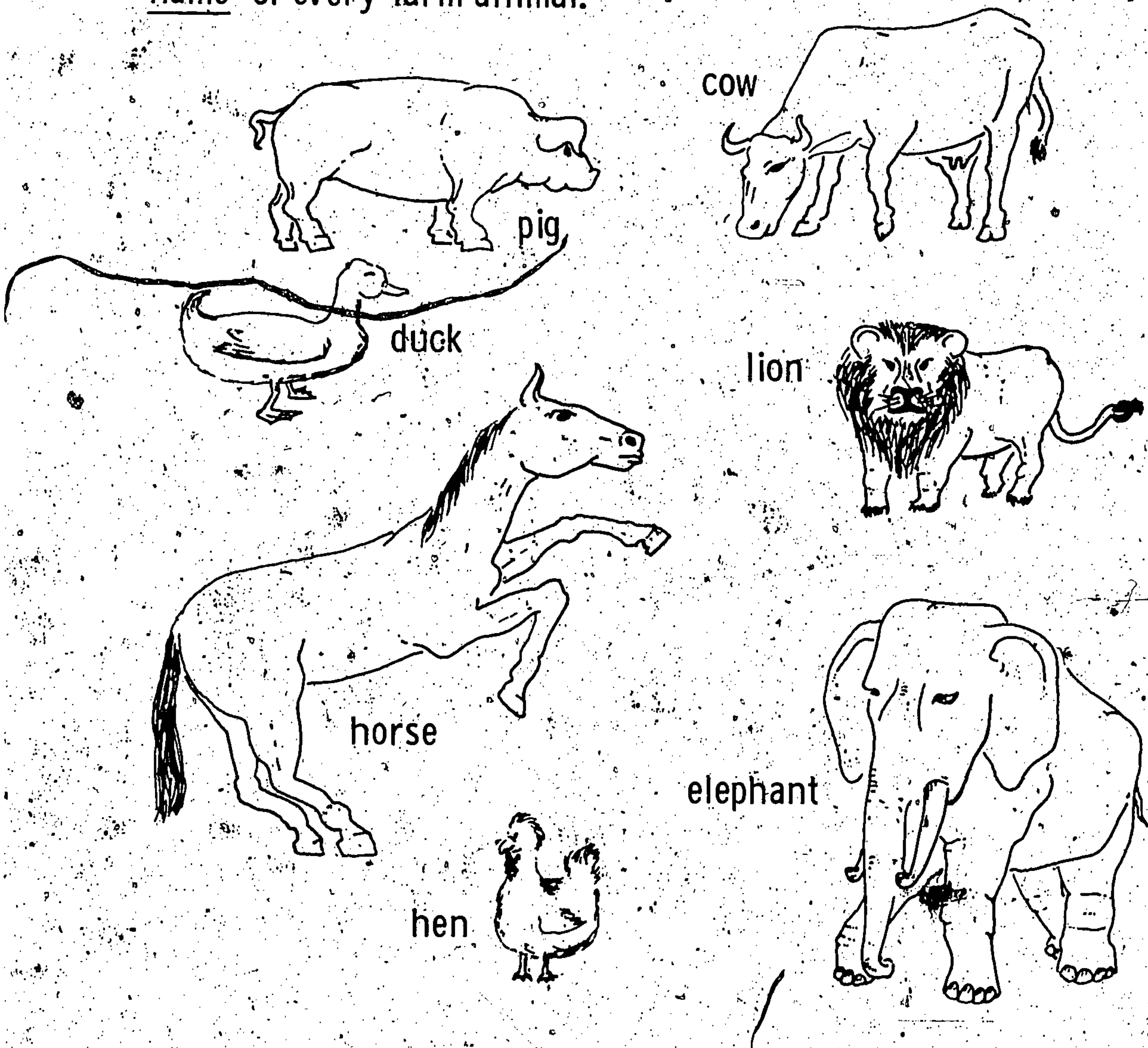
Spinner and blocks.

Team A	Team B

Make tally marks

Winner is ____________

These are the names of animals. Draw a red ring around the name of every animal that has four legs. Draw a blue ring around the name of every farm animal.



1. __5__ animals have four legs.
2. __5__ animals are farm animals.
3. __3__ animals are both farm animals and animals with four legs.
4. __7__ animals are either farm animals or animals with four legs or both.

Look at the words below. Draw a red ring around every word that has the letter s in it. Draw a blue ring around every word that has the letter e in it.

ask	be	dress	fall	find
get	green	guess	is	know
let	man	mother	red	see
street	that	us	use	you

1. 8 words have the letter s in them.
2. 11 words have the letter e in them.
3. 5 words have both the letter s and the letter e.
4. 14 words have either the letter s or the letter e or both.

Use a spinner which is $\frac{1}{2}$ red and $\frac{1}{2}$ blue.

Spin twice and record under "First spin" and "Second spin".

Repeat 16 times.

Now look at your record of spins. Put tally marks in the right place under Results.

See the Sample.

Total the columns.

	Spins: First spin Red	First spin Blue	Second spin Red	Second spin Blue	Results: Red & Red	Exactly one Red	No Red	At least one Red
Sample	I			I		I		I
1								
2								
3								
4								
5								
6								
7								
8								
9								
10								
11								
12								
13								
14								
15								
16								
Total								

1. Which result has the greatest number of tallies? at least one Red (Probably
2. Are any two results equal? ________
3. If so, which ones? ________
4. Which result has the least number of tallies? ________
5. If you were to guess before you did the activity, which result would you guess would have the greatest number of tallies? ________
6. If you repeated the activity many times, would you expect "No Red" to have less tallies than "Exactly one Red". yes
7. If you repeated the activity, would you expect the results to be exactly the same? no

LESSON 5

Number of Possibilities

Objective: To introduce the idea of the number of possibilities for a given experiment and to use simple tables to show these possibilities.

Vocabulary: Possibility, table, row, column, (optional) probability.

Materials: Two red blocks, two blue blocks, two bags, checkers, red and blue crayons, small spinners half-red, half-blue, small spinners one-third red, one-third blue, one-third yellow.

Suggested Procedure:

Review the fact that sometimes there are only two possibilities when something happens. When a bag contains only red and blue blocks, the only colors that can be drawn from it are red and blue. If a child spins a checker, it might, of course, stand on edge, but if we say that we won't count that as a spin, the only possibilities are that it will land with one or the other of the two faces on top. It cannot do both.

Recall the games children played in which they had to draw a block from each of two bags. Suggest that they play the game again, but keep score in a new way. Use sets of two teams, called S (Same) and D (Different). A player draws a block from each of two bags (each containing one red and one blue block). If the two blocks are the same color, it is a point for Team S. If they are of different colors, it is a point for Team D. Record the results and ask about the fairness of the game. Is one team as likely to win as the other? (Yes.)

Now suggest that the boys form one team, the girls another, and you, alone, the third. Use the same bags and blocks as before. If a player draws two red blocks, it is a point for the girls. If he draws two blue blocks, it is a point for the boys. If he draws blocks of different colors, it is a point for you. Play the game and have each pupil keep a record on Pupil page 18 of the drawing, as well as of the score. The sample shows how to record a red block drawn from the first bag and a blue block from the second bag. Discuss the children's answers to the four exercises on this page and lead them to discover why you have a better chance to win than either the boys or the girls. Let them use

their own words to explain that there are two possibilities for drawing different colors but only one for drawing two reds and only one for drawing two blues.

Lessons 10 and 11 could follow this lesson if you think this order would be appropriate for your children. Also, when you reach a point in this unit which seems to be about as far as your children can profitably go, you might want to look at later lessons to see if there are ideas which you might adapt so they would be appropriate for your children.

Pupil page 19: This page should be used with the class working together. Explain that each part of a flag may have only one color, so if red is used for half of the first flag it may not be used for the other half. Children may need to be assured that it is not "wrong" if they leave two flags uncolored. On the contrary, there are only two ways to use red and blue without repetition. You may need to explain that different shades or angles of drawing with the same crayon do not represent a different way of making the flag. Compare each flag to a bag containing one red and one blue block. If the red block is taken out first, all that is left is blue, and vice versa.

Pupil page 20: On this page, the same color may be used in both parts of a box, so each box is like two bags, each of which has a red and a blue block. A child might draw a red block from one bag and a red block from the other bag, so a box might have both parts red, etc. See if children can transfer this idea to the game played at the beginning of this lesson -- boys score on 2 blue blocks, girls on 2 red blocks, and the teacher on 1 red and 1 blue block.

Pupil pages 21 and 22: Complete these two pages as a class activity. Children can share the small spinners or you can spin one spinner and the children can tally the results in the chart.

Discuss the answers to the eight questions and then introduce the table at the bottom of the page. Mention that it is an orderly way to show the possibilities of spinning a spinner and tossing (or spinning) a checker. Mention that the left side shows the result of a spin on the spinner and that this will be indicated first in its row. The top of the table shows the side of the checker which is up, and the result is shown second in the column below its name. Always work from left to right and top to bottom in these tables. Call their attention to the first box, RC, which has been filled in. Have children tell what the R stands for and what the

C stands for. The next "box" in the table to look at is red on the spinner and star on the checker. Children write RS in this box to stand for red on the spinner and a star on the checker. Ask which box should be completed next and then have each pupil write BC in this box to represent blue on the spinner and a crown on the checker. Complete the table by writing BS in the last box. Some children will need extra practice in learning how to complete tables. There are two more to do in this lesson. Always work in a definite order while doing this; we have used the left to right and top to bottom sequence in this unit. Show children how easy it is to "read" the four possibilities from this table.

Pupil page 23: Practice in making and interpreting a table is given on this page. Some children may need to spin the one-third red, one-third blue, one-third yellow spinner to help find the possibilities with this spinner and a checker. Again work with the left to right and top to bottom approach in completing this table. In question 3, children can count to see that the table shows there are six possible results of spinning this spinner and tossing a checker. By looking at the table they can also answer question 4. In 4b and 4c you may need to call attention to the word or. It may be helpful if children draw a red ring around the answers in the table for 4b and a blue ring around the answers for 4c.

Pupil page 24: Letters are used for the children's names in this table. Children may need to be assured that there are only 8 possible pairs of leaders even though the list shows places for 10. If desirable, you might make cards with the six names on them and select cards to find different pairs of children, choosing a card from the boys' pile and a card from the girls' pile.

Optional Further Activities - Games

These games can be used to provide drill in addition and subtraction.

Game 1 - Smaller and Difference

This game for two people uses two spinners with dials of 6 equal parts numbered from 1 through 6, or two cubes with faces numbered from 1 through 6. One player is called D (for difference) and the other S (for smaller). The two cubes are tossed (or the two spinners are spun), and the players look at the two numbers. Player D earns one point if the difference between the two

numbers is greater than the smaller of the two numbers. If the smaller of the two numbers is greater than the difference, player S gets one point. If the difference equals the smaller, for example if the numbers are 4 and 2 or 6 and 3, neither player gets a point. Play to see who gets a certain number of points first, for example 15.

After several children have played the game, ask them if it is a fair game. Would they rather be S or D? This table shows that S has the greater chance to win. Squares are drawn around the cases that are ties. Ovals are drawn around the cases where the difference is greater than the smaller number. The rest of the entries in the table represent the cases where S wins a point. Of the 36 cases: there are 6 ties, D wins in 12 cases, S wins in 18 cases. So S wins $\frac{1}{2}$ of the time and D wins $\frac{1}{3}$ of the time.

TABLE OF DIFFERENCES

First Cube \ Second Cube	1	2	3	4	5	6
1	0	$\boxed{1}$	2	3	4	5
2	$\boxed{1}$	0	1	$\boxed{2}$	3	4
3	2	1	0	1	2	$\boxed{3}$
4	3	$\boxed{2}$	1	0	1	2
5	4	3	2	1	0	1
6	5	4	$\boxed{3}$	2	1	0

Game 2 - Sums and Doubles

Two players use the same two spinners or the two cubes in Game 1. One player is called "Sums" and the other "Doubles". Each player tosses one cube (or spins one spinner). "Sums" adds the numbers shown on the two cubes while "Doubles" looks at his cube and doubles that number. If "Sums's" sum is greater than "Doubles's" double of his number, he gets one point. If not, "Doubles" gets the point. No points are given for ties (for example, if both "Sums's" cube and "Doubles's" cube showed 3). Play until one player gets a certain number of points, 15 for example. Pupils can determine after playing this game that it is a fair game. Of the 36 entries in a 6 x 6 table, S wins 15 times, D wins 15 times, and there are 6 ties.

Game 3 - Add or Double Your Score

This game is a variation of Game 2. Now, however, "Sums's" score is the actual sum on the two cubes (or spinners) and "Doubles's" score is the actual double of the number on his cube (or spinner). For example, if "Sums's" cube showed 5 and "Doubles's" cube showed 3, "Sums" gets an 8 for his score

and "Doubles" gets a 6 for his score. Play until one player gets 100 points. This is a fair game also, as pupils can determine after playing it many times.

Games such as those described above can be played by using other devices which are described in the Appendix.

Girls, Boys, and Teacher

For this game you need two bags. Put a red and a blue block in each bag.

The girls are one team and the boys are another. Your teacher will be the third team.

One of the girls starts by drawing a block from each bag. If the two blocks are red, it is a point for the girls. If both are blue, it is a point for the boys. If the two blocks are of different colors, it is a point for your teacher.

Keep a record on this chart.

	First Bag	Second Bag	Score for: Girls	Boys	Teacher
Sample	R	B			1
1.					
2.					
3.					
4.					
5.					
6.					
7.					
8.					
9.					
10.					
		Totals			

1. Who has the highest score? ______
2. If this game lasted 20 turns, which team do you think would win? ______
3. Does each team have an equal chance to win? no
4. Draw a circle around each way your teacher can win:

1st bag	2nd bag
R	R
R	B
B	R
B	B

Girls, Boys, and Teacher

For this game you need two bags. Put a red and a blue block in each bag.

The girls are one team and the boys are another. Your teacher will be the third team.

One of the girls starts by drawing a block from each bag. If the two blocks are red, it is a point for the girls. If both are blue, it is a point for the boys. If the two blocks are of different colors, it is a point for your teacher.

Keep a record on this chart.

	First Bag	Second Bag	Score for: Girls	Boys	Teacher
Sample	R	B			1
1.					
2.					
3.					
4.					
5.					
6.					
7.					
8.					
9.					
10.					
		Totals			

1. Who has the highest score? ______
2. If this game lasted 20 turns, which team do you think would win? ______
3. Does each team have an equal chance to win? no
4. Draw a circle around each way your teacher can win:

1st bag	2nd bag
R	R
R	B
B	R
B	B

Use your red and blue crayons. See how many different ways you can use just these two colors to color the two parts of the boxes. This time you may use the same color in both parts of a box.

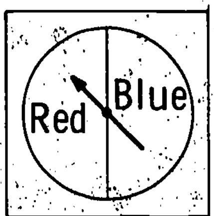
red	red	blue	blue
red	blue	red	blue

1. How many ways did you find? 4
2. How many boxes are all red? 1
3. How many boxes are all blue? 1
4. How many boxes are part red and part blue? 2
5. How many boxes are at least part blue? 3
6. How many boxes are at least part red? 3
7. How many ways can you color these boxes with two colors? 4

Brain Teaser. If you could use three colors, guess how many ways you can color boxes like these. 9 ways to color the 2 parts of the boxes. (For Mary with red, blue + green crayons - RR, RB, RG; BB, BR, BG; GG, GR, GB).

Spinner and Checker

How many possibilities are there if we use this spinner <u>and then</u> toss a checker?



Let's find out.

First, spin the pointer. Put a tally in the chart to show the result. Then spin or toss the checker and tally the result. Do this 12 times.

The sample shows a result of red on the spinner and a star on the checker.

	Spinner		Checker	
	Red	Blue	♔	☆
Sample	1			1
1.				
2.				
3.				
4.				
5.				
6.				
7.				
8.				
9.				
10.				
11.				
12.				

Look at the chart you just made to answer these questions.

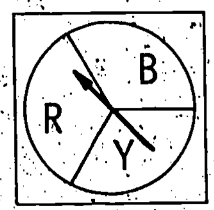
1. In the same turn, did you ever have red on the spinner and a crown on the checker? ______
2. Did at least one of the turns give red on the spinner and a star on the checker? ______
3. In the same turn, did you ever get blue on the spinner and a crown on the checker? ______
4. In the same turn, did the spinner ever show blue and the checker show a star? ______
5. Can you think of any possibility which you did not get? ______
6. How many possibilities are there with this spinner? 2
7. How many possibilities are there with a checker? 2
8. How many possibilities are there with this spinner <u>and</u> a checker? 4

We found four possibilities when we used this spinner and then spun a checker. They are: Red, Crown; Red, Star; Blue, Crown; Blue, Star.

Complete this table using R for Red, B for Blue, C for Crown and S for Star.

Spinner \ Checker	Crown	Star
Red	RC	RS
Blue	BC	BS

Now let's find the possibilities with this spinner and a checker.



Complete this table. You may spin the pointer and toss a checker if you need to.

Checker

		Crown	Star
Spinner	Red	RC	RS
	Blue	BC	BS
	Yellow	YC	YS

1. How many possibilities are there with this spinner? 3
2. How many possibilities are there with the checker? 2
3. How many possibilities are there with this spinner and a checker? 6

4. How many possibilities are there with:
   (a) red on the spinner and a crown on the checker? 1
   (b) red or blue on the spinner and a star on the checker? 2
   (c) yellow on the spinner and a crown or a star on the checker? 2
5. With the spinner of 2 colors and the checker, there are 4 possibilities.
6. With the spinner of 3 colors and the checker, there are 6 possibilities.

Miss Holliday always has one boy and one girl lead the class to lunch. The boys who have not had a turn are Bob, Dan, Tom, and Frank. The only girls who have not had a turn are Mary and Helen.

Miss Holliday writes each name on a card like this:

Bob	Dan	Tom	Frank	Mary	Helen

To find out who will lead the class today, Miss Holliday puts the boys' names in one box and the girls' names in another box. Then she draws one name from each box.

How many different possibilities are there for a boy and a girl to lead the class?

You can complete the table or write the names of the possible leaders here.

Leaders

	Girl	Boy
1.	Mary	Bob
2.	Mary	Dan
3.	Mary	Tom
4.	Mary	Frank
5.	Helen	Bob
6.	Helen	Dan
7.	Helen	Tom
8.	Helen	Frank
9.		
10.		

Boys

Girls	Bob	Dan	Tom	Frank
Mary	MB	MD	mT	mF
Helen	HB	HD	HT	HF

1. With the 2-colored spinner and the checker, there were 4 possibilities.
2. With the 3-colored spinner and the checker, there were 6 possibilities.
3. With 4 boys and 2 girls, there are 8 possible pairs of leaders with one boy and one girl.

LESSON 6

Combinations (of 2 things)

Objective: To count the number of ways two things can be selected from a larger set.

Vocabulary: Combinations.

Materials: None.

Suggested Procedure:

Lessons 6 and 7 are concerned with combinations and Lesson 8 introduces ordered arrangements. You may want to read again sections 7 and 8 of the Mathematical Comments before teaching these lessons. In Lessons 6 and 7 we are obtaining subsets while in Lesson 8 we are working with arrangements in which order is considered, commonly termed permutations. This word need not be used with children.

Present problems similar to the following:

When you play Squirrel in a Tree, you first get into groups of three, and then two of each group join hands to make a tree with the squirrel inside. Suppose Bob, Terry, and Jack are in one group. How many different ways can they make a tree? (Use children in the class and have them demonstrate. Bob and Terry, Bob and Jack, Terry and Jack.)

List the answers in a column with the heading "Trees". Ask how many different combinations were possible.

Pupils	Trees
Bob, Terry, Jack	Bob, Terry
	Bob, Jack
	Terry, Jack

Show children how you did this systematically, going from left to right with the list of three boys, Bob first. Ask children how many different trees can be made if Lee were in the group with Bob, Terry, and Jack. After children have given their predictions, make a new list of "Trees".

Pupils	Trees
Bob, Terry, Jack	Bob, Terry Bob, Jack Terry, Jack
Bob, Terry, Jack, Lee	Bob, Terry Bob, Jack Bob, Lee Terry, Jack Terry, Lee Jack, Lee

Again show how you have followed a definite order in writing the trees that can be made from the four boys. You started with Bob and worked from the list of boys from left to right to find who would be with Bob. Then you moved to the right to Terry and wrote the boys who could be with him, again going from left to right. Finally, you listed Jack and Lee, again following the left to right sequence. Children should learn the importance and advantage of a systematic approach such as is suggested here.

The formula from section 8 of the Mathematical Comments would of course not be used with the children, but it is included here for your interest.

$$\binom{n}{s} = \frac{n!}{s!(n-s)!}$$

Make a chart on the board:

Number in Group	Number of Combinations of 2
3	3
4	6
5	10
6	15

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3\cdot2\cdot1}{2\cdot1\cdot1} = 3$$

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4\cdot3\cdot2\cdot1}{2\cdot1\cdot2\cdot1} = 6$$

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} = 10$$

$$\binom{6}{2} = \frac{6!}{2!(6-2)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot4\cdot3\cdot2\cdot1} = 15$$

Change the context of the problem by suggesting that 5 girls want to jump rope and that they will take turns holding the rope, two at a time. What combinations can be made? List them. On the chart, enter the number in the group (5) and the number of combinations (10). Then include one more child in the group.

Perhaps by this time somebody will be able to see a pattern and to predict the number of combinations. For seven in the group, there are 21 combinations.

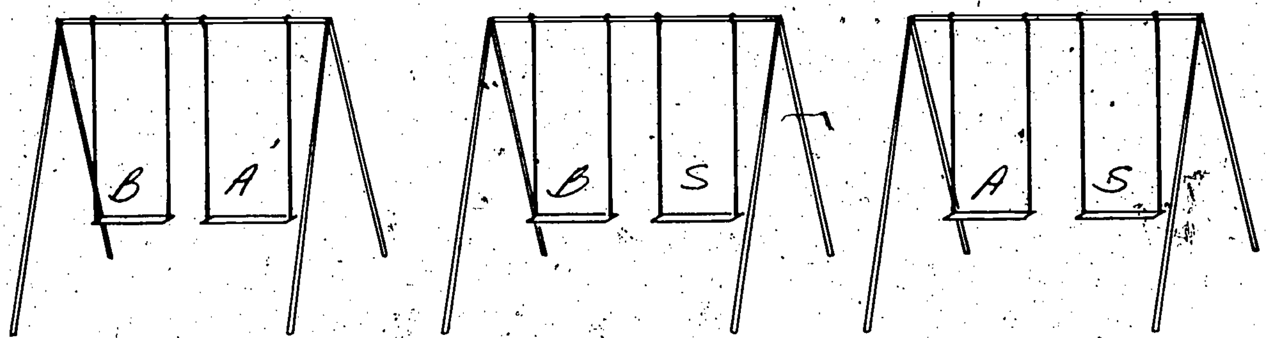
Pupil page 25: Children should have no difficulty with this page. The order of children on the swing or on the teeter-totter does not make a different set. (Order will be considered in Lesson 8.) However, do encourage a systematic way of listing, as Betty and Anne, Betty and Sally, Anne and Sally.

Pupil page 26: The placing of answer lines is such that an orderly approach is encouraged; the teams with Mary first are in column one, Tom first in the second column, and then Susan first in the last column. A left-to-right sequence is followed in writing the names as they are originally stated.

Pupil pages 27 and 28: Again these pages are arranged so that a systematic listing is encouraged. Question 2 on page 28 reminds the pupil of the importance of organization.

25

Betty, Anne, and Sally want to play on a swing that is made for two children. Use B, A, and S for the names of the children. Put these letters on the swings to show the three pairs of children.



From three children we can choose ________ pairs.

Jim, Andy, and Erik want to play on a teeter-totter. Show the different pairs of children (one on each end). Use J, A, and E for the names of the children.

3 pairs

1. How many times does Erik get to play? __2__
2. Three children can teeter-totter (two at a time) in __3__ ways.

Mary, Tom, Susan, and Billy are playing a game. They want to choose a team with 2 children. Show all the teams they can choose. Use M, T, S, and B for their names.

M and T — T and S — S and B

M and S — T and B

M and B

1. How many teams would Tom be on? 3
2. How many teams would Susan be on? 3
3. Each pupil would be on how many teams? 3
4. From 4 children we can choose a team of two in 6 different ways.
5. From 3 children we can choose a team of two in 3 different ways.
6. From 2 children we can choose a team of two in only 1 way.
7. Guess how many teams of two can be chosen from 5 children. 10

In the Brown family, five children -- Mary, Steven, Becky, Linda, and Ricky -- take turns doing dishes. Two children work together each day. Show all the ways they might work. Use M, S, B, L, and R for the names of the children.

M and S	S and B	B and L	L and R
M and B	S and L	B and R	
M and L	S and R		
M and R			

1. Five children can work in pairs in 10 different ways.
2. How many times was Mary listed? 4
3. How many times was Steven listed? 4
4. Each child was listed 4 times.
5. Who was listed with Mary? S B L R
6. Who was listed with Steven? B L R M
7. In 10 days how many times does Linda wash dishes? 4
8. How many different children does Linda wash dishes with in 10 days? 4

In Cedar Falls there is a cub pack which has a den of 6 boys -- John, Chuck, Perry, Andy, George, and Herman. They work together on various projects in teams of two. Show all the ways they might make up a team of two. Use J, C, P, A, G, and H for their names.

J and C	C and P	P and A	A and G	G and H
J and P	C and A	P and G	A and H	
J and A	C and G	P and H		
J and G	C and H			
J and H				

1. Each cub is listed how many times? 5
2. Have you organized your work so that it is easy to find results? ______
3. From 2 children we can choose a team of two in only 1 way.
   From 3 children we can choose a team of two in 3 ways.
   From 4 children we can choose a team of two in 6 ways.
   From 5 children we can choose a team of two in 10 ways.
   From 6 children we can choose a team of two in 15 ways.
4. Guess. From 7 children we can choose a team of two in 21 ways.

LESSON 7

Combinations (of 3 things or 4 things)

Objective: To count the number of ways three or four things can be selected from a larger set.

Vocabulary: (No new terms.)

Materials: None.

Suggested Procedure:

Review briefly Lesson 6 on making combinations of two members from a set. Discuss the possibility of making combinations of more than two. Present a problem similar to the following:

> In a playroom there is a reading table at which only three children can sit. If Pat, Anne, Ben, and Mark are in the playroom, how many different groups can go to the table? (4 . Use children from the class. Pat, Anne, Ben; Pat, Anne, Mark; Pat, Ben, Mark; Anne, Ben, Mark.)

List the possible combinations and make a chart showing the number in the group and the number of combinations of 3 . (Just as a matter of interest to you, the computation by formula is given here.)

Number in Group	Number of Combinations of 3
3	1
4	4
5	10

$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = \frac{3\cdot2\cdot1}{3\cdot2\cdot1\cdot1} = 1$$

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot1} = 4$$

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1} = 10$$

> Suppose Sue comes into the playroom, too. Now how many different groups can go to the table? Do you think we'll have more combinations than we had when there were only 4 children to share the table?

Lead the children to list groups in an orderly way. For example,

Pat, Anne, Ben
Pat, Anne, Mark
Pat, Anne, Sue
Pat, Ben, Mark
Pat, Ben, Sue
Pat, Mark, Sue

Anne, Ben, Mark
Anne, Ben, Sue
Anne, Mark, Sue

Ben, Mark, Sue

How many children would be in the room if there were only one possible combination? (3)

(<u>NOTE</u>: Do not continue unless pupil interest warrants it.)

If there were 6 children in the room, how many combinations of 3 could we find? (20)

You may want to use an illustration with combinations of four things from a set of five and from a set of six before going to the pupil pages. If so, use small objects to show this.

<u>Teacher Note</u>: Even adults have difficulty keeping track of combinations selected from a set. One method which may be helpful to you is to make a chart. For example, let's say we want to get all the possible subsets of 3 from this set of 5 - A, B, C, D, E.

Write the set.

1. Start with the usual method of the first 3, ABC, and make a tally under the letters. Thus:
2. Pick the next group of 3, working from left to right, and tally.
3. Pick the next group of 3 and tally.
4. Pick another group of 3, going from left to right (no more start with AB so go to AC).
5. Tally the next group of 3.
6. Work from left to right again.
7. There are no more groups beginning with A, so now start with B and tally.
8. Tally another group of 3 in left to right order.
9. Tally the next group of 3.
10. Tally the last group of 3.

	A	B	C	D	E
1.	/	/	/		
2.	/	/		/	
3.	/	/			/
4.	/		/	/	
5.	/		/		/
6.	/			/	/
7.		/	/	/	
8.		/	/		/
9.		/		/	/
10.			/	/	/

A look at the chart shows that there are six tallies under each letter. Now the ten groups of 3 can be written for this chart.

Pupil pages 29-31: The first two pages work with combinations of 3 and the next page with combinations of 4. Each of these pages is printed so that the pupil can write the combinations in an orderly way.

Pupil pages 32 and 33: These pages show a way to obtain the combinations from a larger set by building from previous combinations. You will want to do this as a class activity. Some children may prefer to use a different method of obtaining the different combinations and this, of course, is acceptable. The importance of order and a systematic approach is evident in this lesson.

Betty likes all kinds of animals. Her mother will let her have three pets. However, Betty must choose from a dog, bird, fish, and turtle.

Show all the combinations of three pets that Betty can have. Use the letters d, b, f, and t.

1. d b f
2. d b t
3. d f t
4. b f t

1. Did you find the four different combinations of pets? yes
2. A dog is in 3 of these combinations.
3. A bird is in 3 of these combinations.
4. Do not look at your list of combinations. Do you know how many of these combinations include a fish? yes, 3
5. From 4 pets, Betty can choose 4 combinations of 3.
6. From 4 different crayons, how many combinations of 3 could you choose? 4

Betty's mother thinks that a snail might be a satisfactory pet. She lets Betty choose three pets from a dog, bird, fish, turtle, and snail. Show all the combinations of three pets that Betty can choose from these five pets. Use the letters d, b, f, t, and s.

1. d b f
2. d b t
3. d b s
4. d f t
5. d f s
6. d t s
7. b f t
8. b f s
9. b t s
10. f t s

1. Did you write your answers in an organized way? ______
2. From 5 pets, Betty can choose 10 combinations of 3.
3. How many times is a snail in one of the combinations? 6
4. Betty's father puts a penny, nickel, dime, quarter, and half dollar on the table. He asks Betty to make as many different combinations of 3 coins as she can. How many combinations of 3 coins can Betty make from these 5 coins? 10
5. If you could keep just one combination of the coins, which would you choose? D Q H

Betty's mother thinks that a snail might be a satisfactory pet. She lets Betty choose three pets from a dog, bird, fish, turtle, and snail. Show all the combinations of three pets that Betty can choose from these five pets. Use the letters d, b, f, t, and s.

1. d b f
2. d b t
3. d b s
4. d f t
5. d f s
6. d t s
7. b f t
8. b f s
9. b t s
10. f t s

1. Did you write your answers in an organized way? ______
2. From 5 pets, Betty can choose 10 combinations of 3.
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1. d b f
2. d b t
3. d b s
4. d f t
5. d f s
6. d t s
7. b f t
8. b f s
9. b t s
10. f t s

1. Did you write your answers in an organized way? ______
2. From 5 pets, Betty can choose 10 combinations of 3.
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5. If you could keep just one combination of the coins, which would you choose? D Q H

Fred joins the group. They now have Ann, Betty, Charles, David, Elsie, and Fred. The 6 children decide to make up teams of 4. How many different teams of 4 can be formed from 6 children?

We can use the same method as before when Elsie joined the group. Add Fred to each of the "3-child" teams.

A, B, E, F	B, C, E, F	C, D, E, F	A, B, C, F
A, C, E, F	B, D, E, F		A, B, D, F
A, D, E, F			A, C, D, F
			B, C, D, F

Find the remaining teams, using A, B, C, D, E, which do not include Fred.

A, B, C, D A, B, C, E A, B, D, E A, C, D, E B, C, D, E

10 teams have Fred and 5 do not. There are 15 teams in all.

6 children can choose 15 different teams of 4 members.

Fred joins the group. They now have Ann, Betty, Charles, David, Elsie, and Fred. The 6 children decide to make up teams of 4. How many different teams of 4 can be formed from 6 children?

We can use the same method as before when Elsie joined the group. Add Fred to each of the "3-child" teams.

A, B, E, F	B, C, E, F	C, D, E, F	A, B, C, F
A, C, E, F	B, D, E, F		A, B, D, F
A, D, E, F			A, C, D, F
			B, C, D, F

Find the remaining teams, using A, B, C, D, E, which do not include Fred.

A, B, C, D A, B, C, E A, B, D, E A, C, D, E B, C, D, E

10 teams have Fred and 5 do not. There are 15 teams in all.

6 children can choose 15 different teams of 4 members.

List on the board: John, Bob, Mary. Continue to list as the discussion proceeds.

> Suppose John starts, but we go around the ring the other way. Then John is first, Mary is second, and Bob is third. Do we have to start with John? (No.) Let's start with Mary this time.

Make a chart showing the number of different ways sets of 2, 3, and 4 can be ordered. Leave room for a set of 1 at the top, and insert it at the close of the discussion. Just for your interest the computations are shown below as discussed in section 8 of the Mathematical Comments.

Number in Set	Number of Ways of Ordering
2	2
3	6
4	

$2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

Call attention to the difference in the numbers of ways of ordering. You may want to ask children to guess or predict how many arrangements can be made from a set of 4.

Use chalk of 4 different colors to make tally marks on the chalkboard or use crayons on construction paper where all can see.

> Suppose we see how many different ways we can arrange these colors if we start with red. (Make marks as you proceed, one set of four under the other.) We can use red first, and then what? Then any of the other colors.
>
> Let's show red, then green. What color do we want next? If we want yellow third, we will put blue last; but if we want blue third, we will put yellow last. So with red first and green second, we have two arrangements:
>
> R G Y B
>
> R G B Y
>
> Suppose we use red first and yellow second. What can we do?

Continue to list the arrangements using red first. (Notice that a left to right sequence is used, a systematic or orderly approach again!)

R G Y B
R G B Y
R Y G B
R Y B G
R B G Y
R B Y G

Ask questions such as:

How many ways are there in which red is first? red is first and yellow is second? red is first and blue is second? red is first and yellow is third?

Point out the fact that this is just a list of arrangements in which red is first. Ask questions to lead to the idea that each of the other colors, if used first, would also give 6 arrangements. Hence 4 × 6 or 24 arrangements can be made. You may want children to help complete the chart.

Pupil pages 34-36: These three pages give practice in listing arrangements. Spaces are provided so that a systematic approach to listing can be made. A planned procedure is most helpful on page 36. This page should be omitted for some pupils.

Alice and Jane are playing "Follow the Leader". Use A and J to show the ways the line might look.

First A, then J.

First J, then A.

Two children can make 2 arrangements.

Alice and Jane let Sally play "Follow the Leader" with them. Use A, J, and S to show how the line might look.

A	J	S
A	S	J
J	A	S
J	S	A
S	A	J
S	J	A

1. How many times was Alice 1st? 2
2. How many times was Alice 2nd? 2
   3rd? 2
3. How many ways can the 3 children line up? 6
4. As arrangements grow larger, it is hard to keep track of the information. Is yours organized? ____

Miss Johnson has three reading groups. She calls them to study in different orders. One day she called Group 1, then Group 2, and last Group 3. Show all the ways she can arrange the order of the groups.

1, 2, 3   2, 1, 3   3, 1, 2

1, 3, 2   2, 3, 1   3, 2, 1

3 groups can be arranged in 6 ways.

If Miss Johnson had only two reading groups, how many ways could she arrange them? 2

Miss Armstrong likes to vary the order in which she eats her lunch. She always brings a sandwich, some fruit, and a piece of cake. Show the ways she can eat these in different order. Use S, F, and C for sandwich, fruit, and cake.

S, F, C   F, S, C   C, S, F

S, C, F   F, C, S   C, F, S

If she eats the fruit first, what could she eat second? sandwich or cake.

If she eats the fruit first and sandwich second, what might she eat last? cake

Miss Peterson's 3rd grade has 4 favorite games. They often vote to decide which game to play first. Help Miss Peterson by listing the different arrangements in which the games can be played. Use 1, 2, 3, 4 for the names of the games.

1, 2, 3, 4	3, 1, 2, 4
1, 2, 4, 3	3, 1, 4, 2
1, 3, 2, 4	3, 2, 1, 4
1, 3, 4, 2	3, 2, 4, 1
1, 4, 2, 3	3, 4, 1, 2
1, 4, 3, 2	3, 4, 2, 1
2, 1, 3, 4	4, 1, 2, 3
2, 1, 4, 3	4, 1, 3, 2
2, 3, 1, 4	4, 2, 1, 3
2, 3, 4, 1	4, 2, 3, 1
2, 4, 1, 3	4, 3, 1, 2
2, 4, 3, 1	4, 3, 2, 1

How many arrangements are there of the 4 games? 24

LESSON 9

Arrangements and Probability

Objective: To relate arrangements to probability.

Vocabulary: (No new terms.)

Materials: Three cards, 3 in. by 3 in., plain on one side, with the letter P on one card, A on the second, and T on the third; blocks of different colors; bags; red, green, and blue colored chalk or sheets of red, green, and blue construction paper.

Suggested Procedure:

Divide the class into two teams and make a chart on the chalkboard to record scores:

Team A	Team B	No Score

Put the three cards on a table, face down, and scramble them. Have a child turn them over, one at a time. Write the letters on the chalkboard in the order in which he turns them over. Discuss whether or not the letters in that order spell a real word. If they do, it is a point for the player's team. If they do not spell a real word, enter a tally in the "No Score" column.

When several children have had an opportunity to play, list the possible arrangements using the letters P, A, and T. Discuss the arrangements that are real words. Point out that of the six arrangements, three are real words and three are not (PTA is not a word!). Therefore, there are three chances out of six of getting a real word. Play the game until all children have had a turn, and compare the numbers of points in the three sections of the chart.

Show the arrangements of letters in two columns:

PAT	PTA
TAP	TPA
APT	ATP

If your class has had experience in using rational numbers, you will want to call attention to the fact that a probability of $\frac{3}{6}$ is the same as a probability of $\frac{1}{2}$. If, however, the class has not yet encountered rational numbers, simply lead children to see that there is just as good a chance of getting a word as not, so that the two events are equally likely. Compare this with the chance of getting a crown when spinning a checker, of drawing a red block from a bag containing one red block and one blue block or three of each, or of getting red on a half-red, half-blue spinner.

Ask whether it would be fair for Team A to get a point when a real word is spelled and Team B to get a point when no real word is spelled. (Yes.)

Suggest that the letters on the cards be changed to "D", "A", and "N". Would the game still be fair if Team A got a point for a real word and Team B scored when no word was formed? (No.) Allow time for discussion. It is hoped that someone will suggest listing the possible arrangements of D, A, and N, and will point out that the chances for Team A would be 2 out of 6 or 1 out of 3, while Team B would have 4 out of 6 chances or 2 out of 3.

Further Activities:

Draw 2 boxes on the chalkboard and print, as shown:

red	blue

Under each, on the chalk ledge, place a block of the corresponding color.

Tell a story about a first grade class in which the children were just learning to read. The teacher had used word cards and colored blocks to help children associate the words with their meanings. However, one of the kindergarten children came in after school, and, after playing with the blocks, put them back under the words. Since he didn't know which word was which, he just guessed at the place each block should go. How likely was it that he got the right block under each word? Suggest that an experiment might give some idea of the probability of the child's being right.

Have the children work in teams of two. Each pair needs 1 red and 1 blue block, a bag or box, and a piece of paper on which to keep a record. The record sheet should be headed "Red-Blue" and divided into 2 sections headed "Yes" and "No".

Red-Blue	
Yes	No

Both blocks are put into the bag and drawn out, one at a time, without replacement. If the red block is drawn first, we pretend that the kindergartener puts it under the first card, which says red, so a tally is put under "Yes". If the blue is drawn first, a tally is put under "No". Each team should make 20 trials.

Discuss results. Children should realize that there is 1 chance in 2 (probability $\frac{1}{2}$) that the kindergartener guessed correctly. Also, he could not get 1 block under the right word without getting both under the right words.

Pupil page 37: Now suggest that the class can do an experiment which is like the kindergartener playing with three different colored blocks and having three word-cards -- red, blue, and green.

Have pupils use their red, blue, and green crayons to color the boxes on page 37. They should do this in a random way as explained above the boxes. After all of them have colored the three boxes, use the colored chalk or the sheets of construction paper to show three colored boxes in a row on the board, for example blue, red, green. Comment that this is one way the three boxes might be colored and ask if any children colored their boxes in this order.

Make a chart on the board and have each child tell the way he colored the three boxes. Quickly record this with letters, for example:

Blue	Red	Green	Number Right
G	R	B	1
R	G	B	0

Discuss the results. How many times were all 3 in the right order? How many times was 1 in the right place? Was there any time when exactly 2 blocks were in the right places? The chance of having all three in the right places seems to be about 1 in how many? (1 in 6; probability $\frac{1}{6}$.) The chance of having them all wrong seems to be about 1 in how many? (1 in 3.)

Pupils then answer the four questions at the bottom of page 37.

Pupil page 38: This page is related to page 37. Children use their three colors to show all the possible arrangements of the three blocks. Encourage children to use an orderly procedure in coloring the blocks; for example, start with red for the first two rows, start with blue for the next two rows, and with green for the last two rows.

You may want to have the children list the permutations (you do not need to use this word with the children!) involved in this experiment. Ask how many different ways the blocks could have been arranged, and list the ways. For instance, in the case of 3 colors -- red, blue, and green -- the permutations are:

R B G
R G B
B R G
B G R
G R B
G B R

Call attention to the fact that there are 6 possible arrangements; therefore, the chance of having all in the right order is 1 in 6 (probability $\frac{1}{6}$). However, there are only 2 of the arrangements in which none of the colors is in its right place; so the chance of having none right is 2 in 6 (or 1 in 3; probability $\frac{1}{3}$). What may surprise children is that there are 3 chances in 6 of having 1 right (probability $\frac{1}{2}$). Of course, there is no chance of having exactly 2 right, for if 2 are right, the remaining 1 must be right, also.

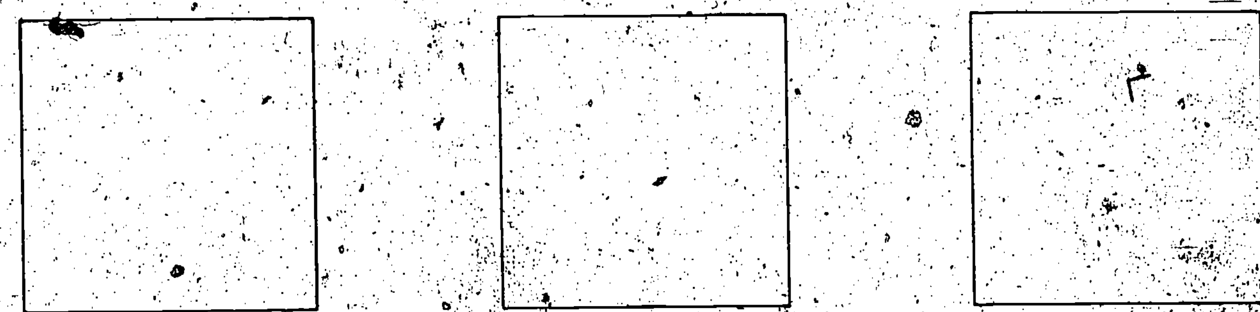
Discuss their answers to the six questions, keeping in mind that the goal is to establish the idea that the chances are 1 out of 6, or the probability is $\frac{1}{6}$ ($P = \frac{1}{6}$), of choosing a particular order of 3 things.

You may want to introduce a fourth color as another class activity and discuss the results. Ask about the number of times all were in the right place, exactly 2 were in the right place, exactly 1 was in the right place, none were in the right place, exactly 3 were in the right place. Here, of course, there will be $4 \cdot 3 \cdot 2 \cdot 1$ or 24 possible arrangements, quite a jump from the 6 possible arrangements with three colors! With 5 colors there would be 120 possible arrangements.

Pupil page 39: This game is like the one which was used to introduce this lesson. Children should write the arrangements in a systematic way.

Match the Boxes

First, color any one of these boxes red. Then color another green. Now color blue the box that has not yet been colored.



Look at the way your teacher has colored the three boxes on the board.

1. Were all three of your boxes in the same place as the teacher's boxes? ________

2. Was just one of your boxes in the same place as the teacher's? ________

3. How many children in your class had boxes colored in the same order as the teacher? ________

4. How many children matched only the teacher's first box? ________ Only the second box? ________ Only the third box? ________

Use your red, blue, and green colors. Show all the possible arrangements of the three boxes.

1.	R	B	G
2.	R	G	B
3.	B	R	G
4.	B	G	R
5.	G	R	B
6.	G	B	R

1. How many times does one of your rows of boxes match your teacher's row? 1
2. How many chances in six are there that one of your rows will match your teacher's row? 1 for any 1 row 6 chances in 6 that 1 of the 6 rows will match the teacher's row.
3. How many times are none of your colors in the same place as your teacher's? 2
4. How many chances are there in six that none of your colors are in the same place as your teacher's? 2
5. How many times do you have just one box match the teacher's boxes? 3
6. How many chances in six are there that just one of your boxes will match the teacher's boxes? 3

A BIRTHDAY PARTY GAME

Mrs. Black made a game for Tom's friends to play at a party. She put cards with letters on them in four boxes. Each boy chose a box. Then he drew the letters out of his box, one by one, without looking. Mrs. Black wrote the letters down as the boy drew them. If the letters were in the right order to make a real word, he got a prize. Here are the boxes of letters. Write all the arrangements for the letters in each box under the box.

Blue box	Green box	Pink box	Yellow box
t n e	t r a	f n a	m r a
tne	*tra*	*fna*	*mra*
ten ×	*tar* ×	*fan* ×	*mar* ×
nte	*rta*	*nfa*	*rma*
net ×	*rat* ×	*naf*	*ram* ×
etn	*atr*	*afn*	*amr*
ent	*art* ×	*anf*	*arm* ×

1. Put an X to the right of each arrangement that is a real word.
2. Which boxes would you most want to get? *green* or *yellow*
3. Which box would you least want to get? *pink*
4. Which box gives you 2 out of 6 chances to make a word? *blue*
5. In which box is the chance for a word 1 out of 6? *pink*

LESSON 10

Repeated Trials

Objective: To relate arrangements from repeated trials to probability.

Vocabulary: (No new terms.)

Materials: 3 bags, each containing 1 red and 1 blue block; spinners with dials half-red, half-blue.

Suggested Procedure:

> Remember the pages on which you were asked to color parts of flags red or blue? (pages 19 and 20) When we talked about that exercise, we compared the second page, where you could use the same color twice in one flag, to choosing blocks from 2 bags, each containing one red and one blue block. Without using real blocks, let's list all the things that might happen when you choose blocks from 2 bags.

Ask questions which will show that from the first bag a red block might be chosen; in that case, one would get either 2 reds or a red and a blue, depending on what one drew from the second bag. Write:

R R   R B

Repeat, showing the possibilities of drawing a blue block first. Show:

R R   R B   B B
       B R

> How many possibilities are there all together? (4) What is the chance of getting 2 red blocks? (1 out of 4, or $\frac{1}{4}$) Of getting 2 blue blocks? (1 out of 4, or $\frac{1}{4}$) Of getting 1 of each color? (2 out of 4, or $\frac{1}{2}$)

Pupil page 40: Use this page as a class activity. Then lead the pupils to relate their work to the activity of drawing a block from each of the 3 bags, each containing 1 red block and 1 blue block. Make a list of the possibilities

on the chalkboard:

3 Reds	2 Reds	1 Red	No Reds
RRR	RRB	RBB	BBB
	RBR	BRB	
	BRR	BBR	

How many possibilities are there all together? (8) In how many ways can you get all red blocks? (1) In how many ways can you get all blue blocks? (1) What is the chance of getting all red? (1 out of 8, or $\frac{1}{8}$) Of getting all blue? (1 out of 8, or $\frac{1}{8}$) Of getting all one color? (2 out of 8, or $\frac{1}{4}$) Of getting exactly 2 reds? (3 out of 8, or $\frac{3}{8}$) Of getting at least 2 reds? (4 out of 8, or $\frac{1}{2}$) Etc.

Make another chart on the chalkboard:

All red	2 red 1 blue	1 red 2 blue	All blue

Have children take turns drawing a block from each bag, and use tally marks to show the results. Discuss the results in relation to your earlier questions on the chances of getting a given outcome.

Pupil page 41: Do this page as a class activity. The spinners could, of course, be spun consecutively instead of at the same time, or one spinner could be spun three times. It will probably not be necessary to actually spin the spinners in order to determine the arrangements of colors that are possible. Relate the spinning of these three spinners to the activity of drawing a block from each of 3 bags, each of which contains 1 red block and 1 blue block. Ask questions as before about the chances of getting certain results.

Pupil pages 42 and 43: Each pupil spins the half-red, half-blue spinner in sets of 3 spins and tallies the results of each spin under 1st Spin, 2nd Spin, and 3rd Spin. He does this for twenty-four sets of spins. Then in the right-hand portion of the table he marks, in the proper columns, the results

of each set of 3 spins. He then totals these last four columns and answers the questions on page 43. In Question 5, the probability of "No Reds" is $\frac{1}{8}$, so we would expect in the long run that "No Reds" would occur about $\frac{1}{8}$ of the time. Our expectation for 24 trials is for "No Reds" to occur about 3 times. In Question 7, most children will answer "No". It w[illegible] be very unusual to find, in just 24 trials, that $\frac{1}{8}$ of the trials, o[illegible] were "Three Red", $\frac{3}{8}$ of the trials, or 9, were "Exactly Two Reds", $\frac{3}{8}$ of the trials, or 9, were "Exactly One Red", and $\frac{1}{8}$ of the trials, or 3, were "No Reds". If appropriate, however, work with children to help them see that in the long run the expected results would be close to the fractions we have indicated for the probabilities. The Brain Teaser gets at this idea.

Use your red and blue crayons to color the three parts of these flags. See how many different ways you can color them.

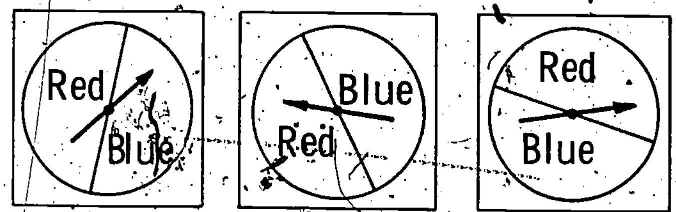
3 Reds	2 Reds	1 Red	No Reds
R	R	R	B
R	R	B	B
R	B	B	B
	R	B	
	B	R	
	R	B	
	B	B	
	R	B	
	R	R	

1. How many ways are there? 8 How many flags are all red? 1
2. How many flags have more red than blue? 4
3. How many flags have more blue than red? 4
4. How many flags are all blue? 1
5. How many flags have at least 1 part red? 7
6. How many flags have at least 1 part blue? 7

On another page we recorded the results of two spins of a $\frac{1}{2}$ red and $\frac{1}{2}$ blue spinner. We noted that it is possible to get:

Two Reds	One Red	No Reds
Red, Red	Red, Blue	Blue, Blue
	Blue, Red	

Here are pictures of 3 spinners.



A B C

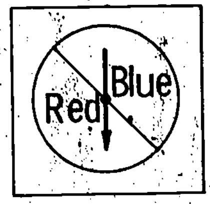
If all three are spun at the same time, list the arrangements of colors we could get.

Three Reds	Two Reds	One Red	No Reds
R, R, R	R, R, B	R, B, B	B, B, B
	R, B, R	B, R, B	
	B, R, R	B, B, R	

1. How many different arrangements are there? 8
2. How many are all red? 1
3. How many have exactly two red? 3
4. How many have at least two red? 4
5. How many have at least one red? 7
6. How many have no reds? 1

Brain Teaser. If you spin the 3 spinners 16 times, you would expect RRR to happen about 2 times.

Use this spinner.



The spinner is to be spun in groups of three spins.

Record the results.

1st spin		2nd spin		3rd spin		Three Reds	Exactly Two Reds	Exactly One Red	No Reds
Red	Blue	Red	Blue	Red	Blue				
					Totals				

Use the totals on the table you just completed to answer:

1. How many "Three Reds" are there? ______
2. How many "Exactly Two Reds"? ______
3. How many "Exactly One Red"? ______
4. How many "No Reds"? ______
5. On 24 groups of spins, about how many "No Reds" would you expect? 3
6. Did you get the same number of "Three Reds" as "No Reds"? ______
7. Are your results exactly the same as the expected results? ______
8. In many groups of spins, would you expect the number of "Three Reds" to be about the same as "No Reds"? yes

Brain Teaser. In 800 groups of spins:

(a) I would expect "Three Reds" to happen about 100 times.

(b) I would expect "Exactly Two Reds" to be the result about 300 times.

(c) I would expect "Exactly One Red" to be the result about 300 times.

LESSON 11

Repeated Trials (Continued)

Objectives: To gain skill in using tables, first introduced in Lesson 5, for organizing data; to use tables for organizing and interpreting data from repeated trials.

Vocabulary: (No new terms.)

Materials: Two cubes of different colors (with a felt pen, number the faces of each cube using numerals 1 through 6 -- or 2 through 7, 3 through 8, etc., depending on the addition practice desired); oatmeal box or other container for the cubes.

Suggested Procedure:

In the preceding lesson, children listed the possibilities for the experiment of drawing 1 block from each of two bags. State that we could find the possibilities systematically, using a table. Show on the chalkboard:

		Second Bag	
		Red	Blue
First Bag	Red		
	Blue		

Remind the children that they first used a table in Lesson 5 and ask one of them to place the entries in this table. You may need to explain again that the left side shows the color of the block taken from the first bag and that color will be indicated first in its row. The top of the table shows the color of the block from the second bag, so that color is written second in the column below its name.

		Second Bag	
		Red	Blue
First Bag	Red		
	Blue		

Show that the possibilities are all given, as they were when you wrote in the last lesson:

RR RB BB
BR

You might say that a table makes it possible to find the possibilities.

Now in order to find the possibilities when a block is taken from each of 3 bags, use another table. On the left, show the possibilities from 2 bags (the entries in the first table). At the top show the possibilities from the third bag, and have children help to complete the table.

Third Bag

First Two Bags		Red	Blue
	RR	RRR	RRB
	RB	RB_	
	BR		_BR
	BB		

Again, rewrite in the form used in the preceding lesson so children can see that no possibility was omitted.

3 Reds	2 Reds	1 Red	No Reds
RRR	RRB	RBB	BBB
	RBR	BRB	
	BRR	BBR	

Use cubes and an oatmeal box or other container for the cubes. Number the faces as suggested in Materials so that your children can get the practice in addition that they need. Have a child shake the cubes in a box and spill them out on a desk. (A few newspapers or a piece of heavy cloth on the desk will help to reduce the noise!) The top faces of the cubes are the scores. Keep a record on the chalkboard. This example is for cubes with faces numbered from 1 through 6.

	Yellow Cube	Green Cube	Total Score
Mary	3	2	5
Jim			
Lucy			

When all have had a turn, have children count to find out how many times each numeral appeared on the yellow cube and on the green cube. Find out how many times each total score was made. Use a chart like the following on the chalkboard:

Yellow Cube	1	2	3	4	5	6

Green Cube	1	2	3	4	5	6

Total Score	1	2	3	4	5	6	7	8	9	10	11	12

Discuss the fact that it was just as likely that on either cube the 3 would be up as that the 5 would, etc.

> How many possibilities are there if just one cube is used? (6)
> Are all 6 possibilities equally likely? (Yes)

Compare with the results shown on the chart for each cube alone.

> How many possibilities are there for total scores? (11)
>
> Which score was made most often? Which score was made least often?
>
> When we added the two numbers on the faces that were up we got the sums, which were the total scores. How many arrangements do you think would give a sum of 2 ? (1) Of 7 ? (6) Of 11 ? (2)

Ask children to suggest a simple way of finding all possible arrangements. Remind them of the tables used before, if necessary, and encourage them to tell how a table for the new problem should be made. Show the table on the chalkboard:

Green Cube (columns); Yellow Cube (rows)

	1	2	3	4	5	6
1	2	3	4		6	
2	3	4				
3						
4						
5						
6						

Let children help to complete the table, and then repeat earlier questions about combinations whose sum is 2, 7, etc.

> If you have 36 different possibilities, what are the chances of getting a score of 6 ? (5 out of 36, or $\frac{5}{36}$) Of 2 ? (1 out of 36, or $\frac{1}{36}$) Etc. What score do you have the best chance to get? (7) What scores are you least likely to get? (2, 12)

Show the "1" in the chart for total scores made when the cubes were actually tossed.

> Nobody made a score of 1 . Why? (Impossible.) What chance did you have of making a score of 1 ? (0) Of 13 ? (0) Of 2 ? (1 out of 36, or $\frac{1}{36}$)

(If you have used rational numbers, show the probabilities as 0, $\frac{1}{36}$, etc.)

<u>Pupil pages 44-46</u>: These pages provide practice in using tables to organize data and present opportunities to interpret these data. You may want to use rational numbers and bring out, for example in Exercise 3 page 44, that the probability of getting red on both spins is $\frac{1}{4}$. It would be easy to move from "chances" to rational numbers if your children have the background.

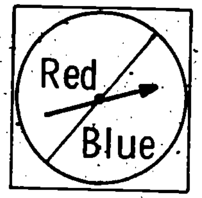
<u>Pupil page 47</u>: This is the first page on which two tables are built. You may want to complete it as a class activity. The 3 spins on the spinner can be compared to drawing a block from each of 3 bags each of which contains a red and a blue block. It can also be compared with page 41, the three

spinner page, and with pages 42 and 43 in which an actual experiment was performed.

Pupil pages 48 and 49: Encourage children to list the possibilities in Exercise 3 in an orderly manner, that is, to complete this exercise by reading from the table in a left to right and top to bottom sequence. Discuss the answers with the children; making additional explanations as needed. For example, for two spins of this 3 colored spinner there are 3×3 or 9 possibilities, while for 3 spins of this spinner there are $3 \times 3 \times 3$ or 27 possibilities. Exercise 14 asks for the number of possibilities for 4 spins. You may want to ask the same question for 5 and 6 spins.

Pupil page 50: Children should first complete the left side of the table and then make the 64 entries in the table. Discuss this page as you did pages 48 and 49.

Pupil pages 51 and 52: On page 51, the pupil builds three tables and uses the data from these to answer questions. Questions 7 and 8 go with the table form on page 52. You may want to do these pages as a class activity, having children show how they obtained the answers to each question.



Write in the table to show all the possibilities of two spins on the spinner. Red and blue are equally likely.

Second spin

First spin	Red	Blue
Red	R R	R B
Blue	B R	B B

1. This table shows there are 4 different possibilities when we spin this spinner twice.

2. Complete this list of possibilities:

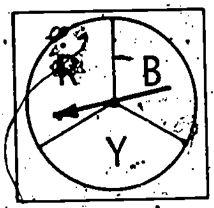
Both Red	Exactly 1 Red	0 Red
R R	R B	B B
	B R	

3. The chances of getting red on both spins are 1 in 4.

4. There is 1 chance in 4 of getting blue on both spins.

5. How many chances in 4 are there of getting only one red on two spins? 2

6. How many chances in 4 are there of getting at least one blue on two spins? 3



This spinner is spun two times. Use the table to show all the possible results. The three colors are equally likely.

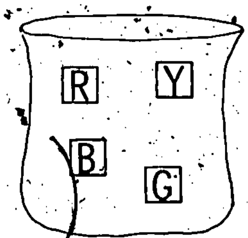
		Second spin		
		R	Y	B
First spin	R	RR	RY	RB
	Y	YR	YY	YB
	B	BR	BY	BB

1. With two spins there are 9 possible results.
2. The chances are 1 in 9 that both spins will stop on red.
3. The chances of blue on both spins are 1 out of 9.
4. Of the 9 possibilities, there are 5 which have at least one yellow.
5. The chances are 5 in 9 that at least one spin will stop on blue.
6. Look at your table. How many times do you have RY ? 1
   The chances of getting RY are 1 in 9.
7. The chances of getting a YY are

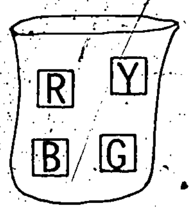
   equal to   less than   greater than

   getting an RB. (Circle the correct one.)

Each of these bags has a red, a yellow, a blue, and a green block. You draw a block from each bag, both bags at the same time. Then each block is returned to its bag. Use a table to show all the possibilities.



Second Bag



First Bag

	R	Y	B	G
R	RR	RY	RB	RG
Y	YR	YY	YB	YG
B	BR	BY	BB	BG
G	GR	GY	GB	GG

1. There are <u>16</u> possible results.

2. The result RR has <u>1</u> chances in <u>16</u>.

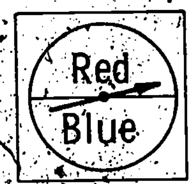
3. Of the 16 possibilities, there are <u>6</u> which have only one green. So there are <u>6</u> chances in 16 of having only one green block in one of the possibilities.

4. The chances of at least one green are <u>7</u> out of <u>16</u>.

5. A result of a blue block and a green block happens <u>2</u> times out of 16. (BG and GB)

6. Will a blue block and a yellow block be expected more often than two yellow blocks? <u>Yes</u> (BY and YB but only 1 YY.)

This spinner is spun three times. Complete the tables to show all the possible results.



Complete this table first.

First spin	Second spin: Red	Second spin: Blue
Red	RR	RB
Blue	BR	BB

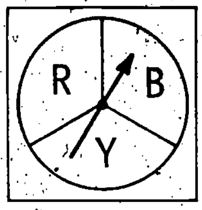
First two spins	Third spin: Red	Third spin: Blue
R R	RRR	RRB
R B	RBR	RBB
B R	BRR	BRB
B B	BBR	BBB

Complete this list of possibilities.

3 Red	2 Red	1 Red	0 Red
RRR	R R B	R B B	BBB
	R B R	B R B	
	B R R	B B R	

1. The tables and the list show there are 8 possibilities when we spin the spinner three times.
2. How many chances in 8 are there of getting blue on all three spins? 1
3. There are 3 chances in 8 of getting exactly two reds on three spins.
4. There are 7 chances in 8 of getting at least one blue on three spins.
5. Which is more likely, A or B? Circle the letter of the correct answer.
   - A. to get two reds and one blue
   - B. to get three blues

Brain Teaser. It is 3 times as likely that the spinner will stop on red exactly twice in 3 spins than on red 3 times in 3 spins.



We found the possibilities when this spinner is spun two times. Our table was:

Second spin

First spin	R	Y	B
R	R R	R Y	R B
Y	Y R	Y Y	Y B
B	B R	B Y	B B

Complete the table below to show all the possibilities of three spins.

Third spin

First and Second spins	R	Y	B
R R	R R R	R R Y	RRB
R Y	RYR	RYY	RYB
R B	RBR	RBY	RBB
Y R	YRR	YRY	YRB
YY	YYR	YYY	YYB
YB	YBR	YBY	YBB
B R	BRR	BRY	BRB
BY	BYR	BYY	BYB
BB	BBR	BBY	BBB

1. When we spin this spinner 2 times, there are 9 possible results.

2. When we spin this spinner 3 times, there are 27 possible results

3. List the possibilities for 3 spins under these headings:

RRR	YYY	BBB	2R	2Y	2B	all different
RRR	YYY	BBB	RRY	RYY	RBB	RYB
			RRB	YRY	YBB	RBY
			RYR	YYR	BRB	YRB
			RBR	YYB	BYB	YBR
			YRR	YBY	BBR	BRY
			BRR	BYY	BBY	BYR

4. How many times does the result have at least 2 reds? 7

5. There are 6 times when the result has exactly 2 reds.

6. There are 6 times when the result has exactly 2 yellows.

7. There are 6 times when the result has exactly 2 blues.

8. There are 6 results which have one red and one yellow and one blue.

9. How many of the possibilities had exactly 1 red? 12

10. How many of the possibilities had exactly 1 yellow? 12

11. Is it twice as likely that a result will have exactly 1 blue as exactly 2 blues? yes

12. Which of these results is more likely?

    3 blues   2 yellows and a red   3 reds   all different   all different

13. On three spins, which of these results are you least likely to get?

    2 blues and a yellow   2 reds and a yellow   3 yellows   all different

    3 yellows

14. How many possibilities are there if we spin this spinner 4 times? 81

Each of these bags has a red, a yellow, a blue, and a green block.

We found the possibilities of drawing a block from each of two bags. They were:

R R	R Y	R B	R G
Y R	Y Y	Y B	Y G
B R	B Y	B B	B G
G R	G Y	G B	G G

How many possibilities are there when we draw a block from each of two of these bags? 16

Now we will think about drawing a block from each of three bags. Complete this table to show all the 3-block arrangements you can make.

First two blocks \ Third block	R	Y	B	G
RR	RRR	RRY	RRB	RRG
RY	RYR	RYY	RYB	RYG
RB	RBR	RBY	RBB	RBG
RG	RGR	RGY	RGB	RGG
YR	YRR	YRY	YRB	YRG
YY	YYR	YYY	YYB	YYG
YB	YBR	YBY	YBB	YBG
YG	YGR	YGY	YGB	YGG
BR	BRR	BRY	BRB	BRG
BY	BYR	BYY	BYB	BYG
BB	BBR	BBY	BBB	BBG
BG	BGR	BGY	BGB	BGG
GR	GRR	GRY	GRB	GRG
GY	GYR	GYY	GYB	GYG
GB	GBR	GBY	GBB	GBG
GG	GGR	GGY	GGB	GGG

1. How many possibilities are there when a block is withdrawn from each of these three bags? 64
2. How many of these possibilities are BBB? 1
3. What are the chances in 64 of getting GGG? 1
4. Is it likely that 2 reds and a yellow will be the result more often than 2 yellows and a red? no
5. How many times can we expect:
   - 2 reds and 1 blue? 3
   - 2 blues and 1 yellow? 3
   - 2 greens and 1 red? 3
   - 2 yellows and 1 green? 3

Jim likes cake and Jello for dessert. He can have just one dessert each day. On Tuesday, Thursday, Friday, and Sunday of one week his mother let him choose which one he wanted. Find all the possibilities for his dessert on those four days. Use C for cake and J for Jello.

A. Tuesday C or J

B. Thursday

Tuesday	C	J
C	CC	CJ
J	JC	JJ

C. Friday

Tuesday and Thursday	C	J
C,C	CCC	CCJ
CJ	CJC	CJJ
JC	JCC	JCJ
JJ	JJC	JJJ

D. Sunday

Tuesday, Thursday, and Friday	C	J
C,C,C	CCCC	CCCJ
CCJ	CCJC	CCJJ
CJC	CJCC	CJCJ
CJJ	CJJC	CJJJ
JCC	JCCC	JCCJ
JCJ	JCJC	JCJJ
JJC	JJCC	JJCJ
JJJ	JJJC	JJJJ

Look at Table B which shows the possibilities for Tuesday and Thursday.

1. In how many ways can he have dessert for these two days? 4
2. Can he have cake on Tuesday and cake on Thursday? yes
3. If he has cake on Tuesday, he can have cake or jello on Thursday.
4. Put an X in this box which stands for Jello on Tuesday and Jello on Thursday.

Thursday

Tuesday	C	J
C		
J		X

Look at Table C which shows the possibilities for Tuesday, Thursday, and Friday.

Tuesday and Thursday	Friday: C	Friday: J
C,C		
C,J		
J,C	O	X
J,J		

5. How many possibilities are there for dessert on these three days? 8

6. There are 4 chances in 8 of having Jello on at least 2 of these days.

7. Put an X in the box which stands for Jello on Tuesday and Friday and cake on Thursday.

8. Put an O in the box which stands for cake on Thursday and Friday and Jello on Tuesday.

Look at Table D which shows the possibilities for all four days.

9. How many possibilities are there for dessert on these four days? 16

10. This is 2 times as many ways as for three days.

11. In how many ways can he have cake twice and Jello twice? 6

12. In how many ways can he have Jello exactly 3 times? 4

13. There are 4 chances out of 16 to have cake exactly 3 times.

14. In how many ways can he have cake at least 1 time? 15

15. In how many ways can he have Jello at least 2 times? 11

16. In how many ways can he have cake more than 3 times? 1

APPENDIX

Probability Devices

This section suggests devices which you might find helpful. It is taken from the Pupil Text of Probability for Intermediate Grades, and so it is written for children. However, from it you may wish to plan activities for supplementary work for individual children or groups.

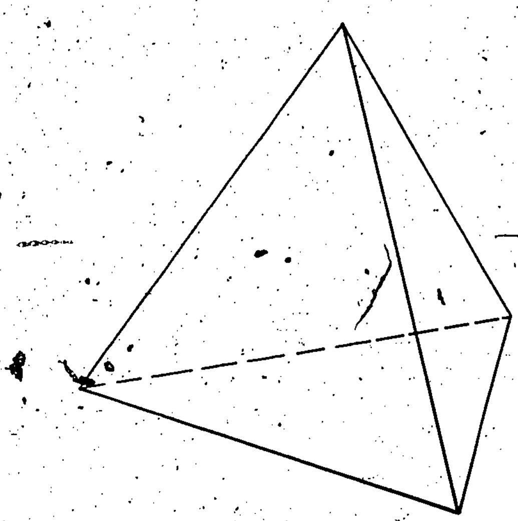
1. Regular Solids

We have used coins, dice, cubes, cards, and other materials to help us learn about probability, but there are many devices which are just as useful. The patterns on the next six pages are for the construction of regular solids which can be used in probability experiments. The patterns can be traced on tagboard and the solids will then be sturdy enough to toss or roll.

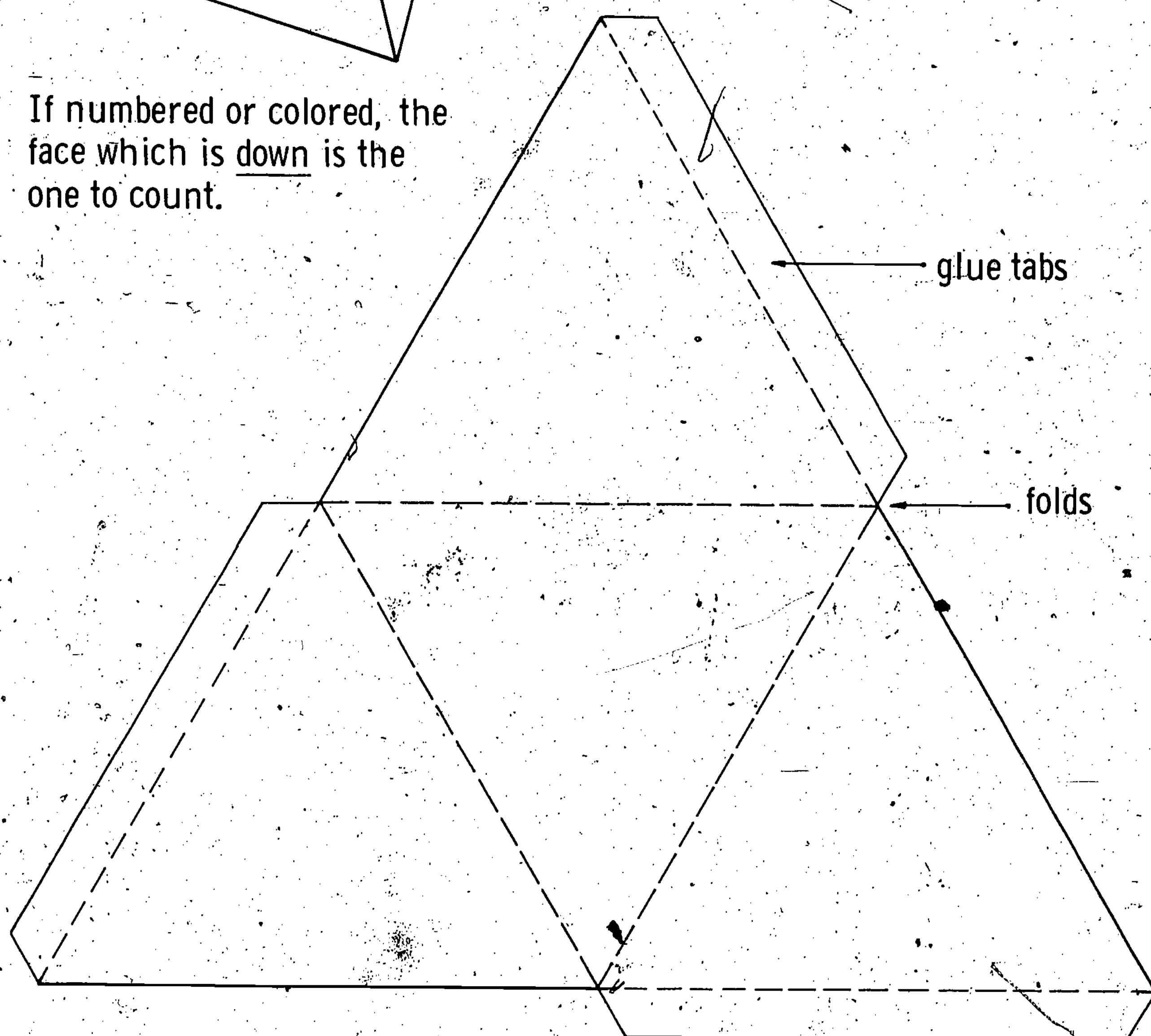
The tetrahedron, octahedron, and hexahedron (cube) are not difficult to construct. Just fold on the dotted lines and glue the tabs.

The dodecahedron is more difficult to construct. Make the first half of it by cutting to the dotted line at each arrow. Then fold on the dotted lines and glue the tabs. Complete by folding the second half and gluing it to the first half, tab by tab. Do not make both halves and then try to put them together.

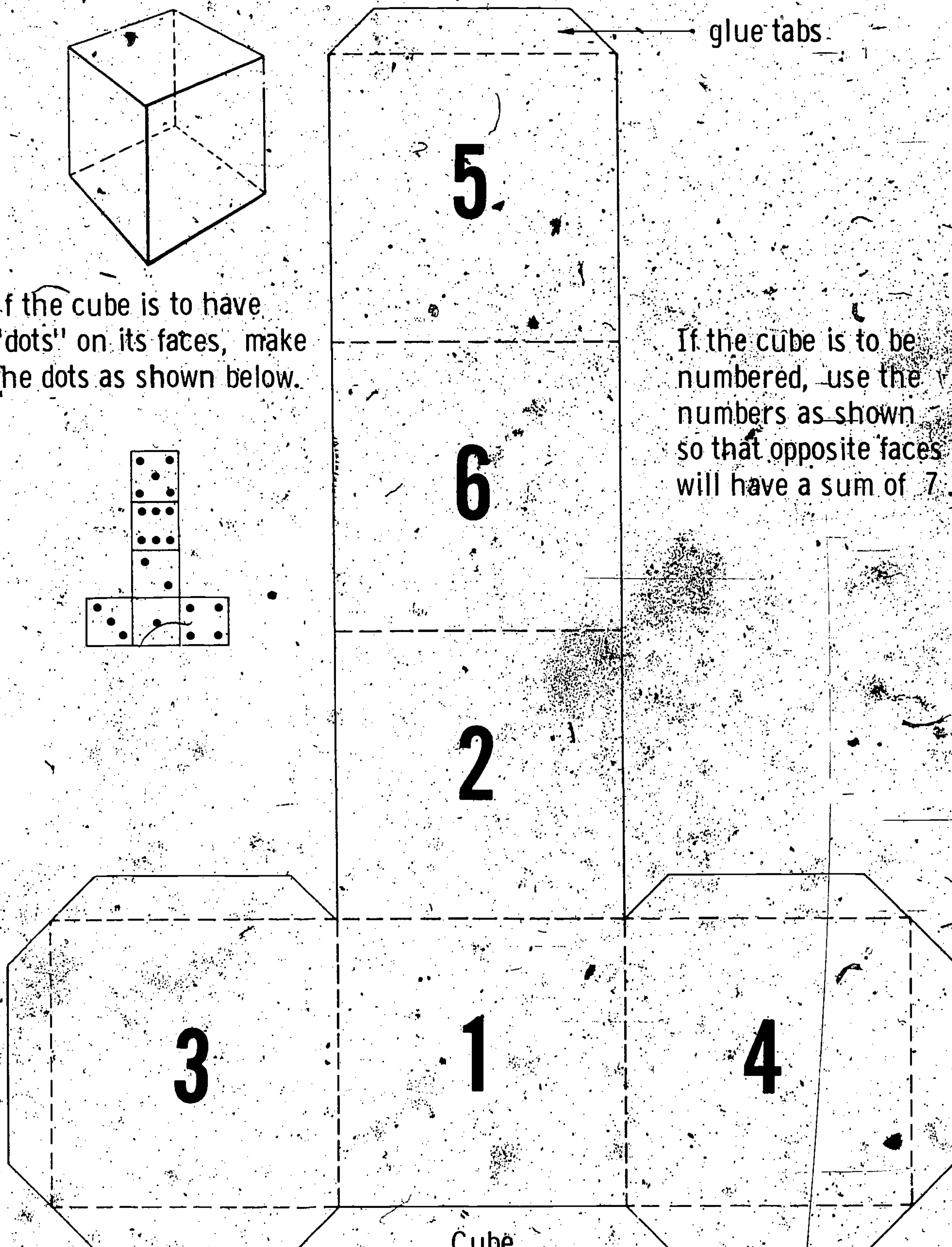
The icosahedron appears difficult to construct, but it isn't. Cut to the dotted line at each arrow. Then fold and glue the tabs in order, one by one, and it will work out nicely.



If numbered or colored, the face which is down is the one to count.



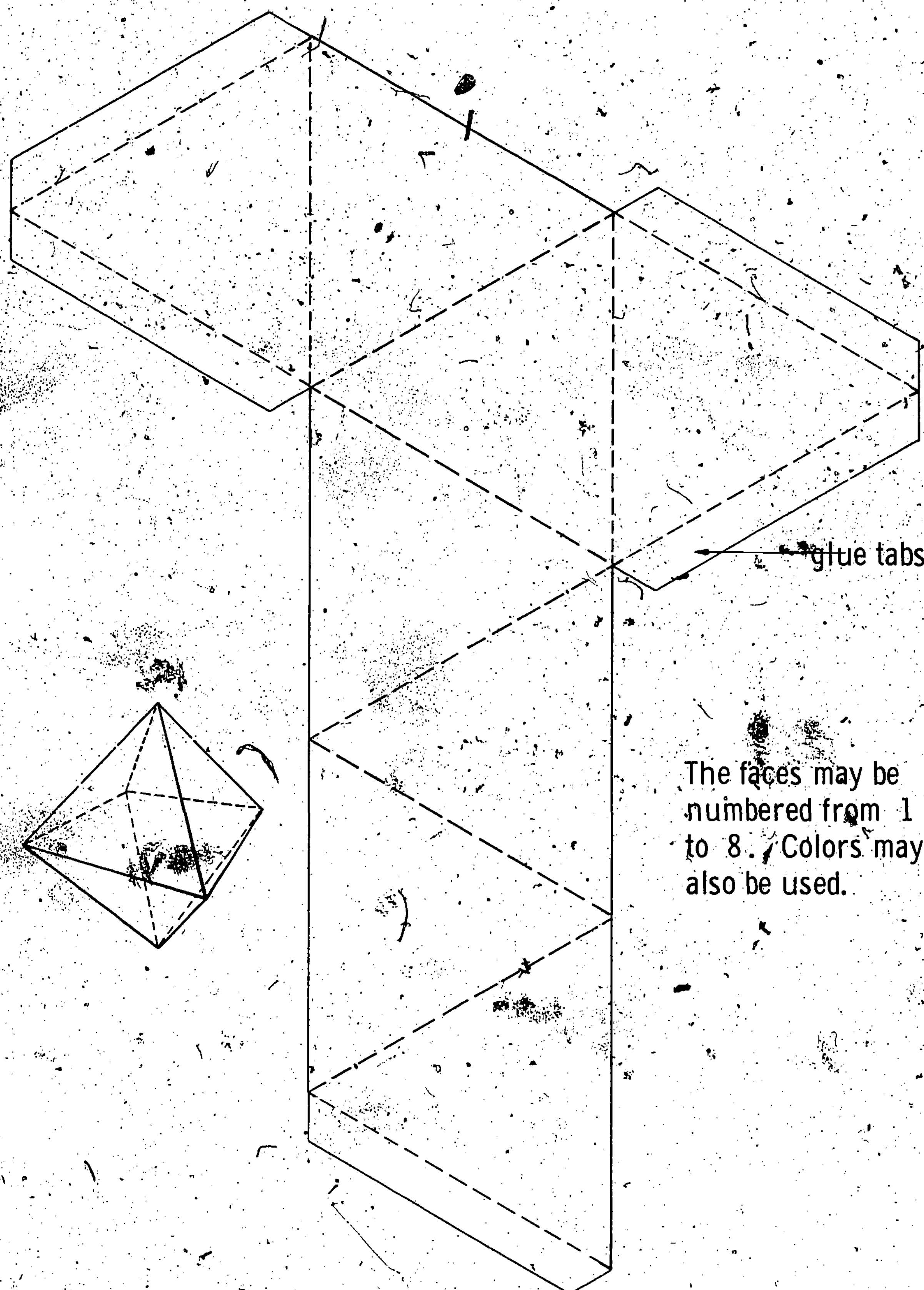
Tetrahedron



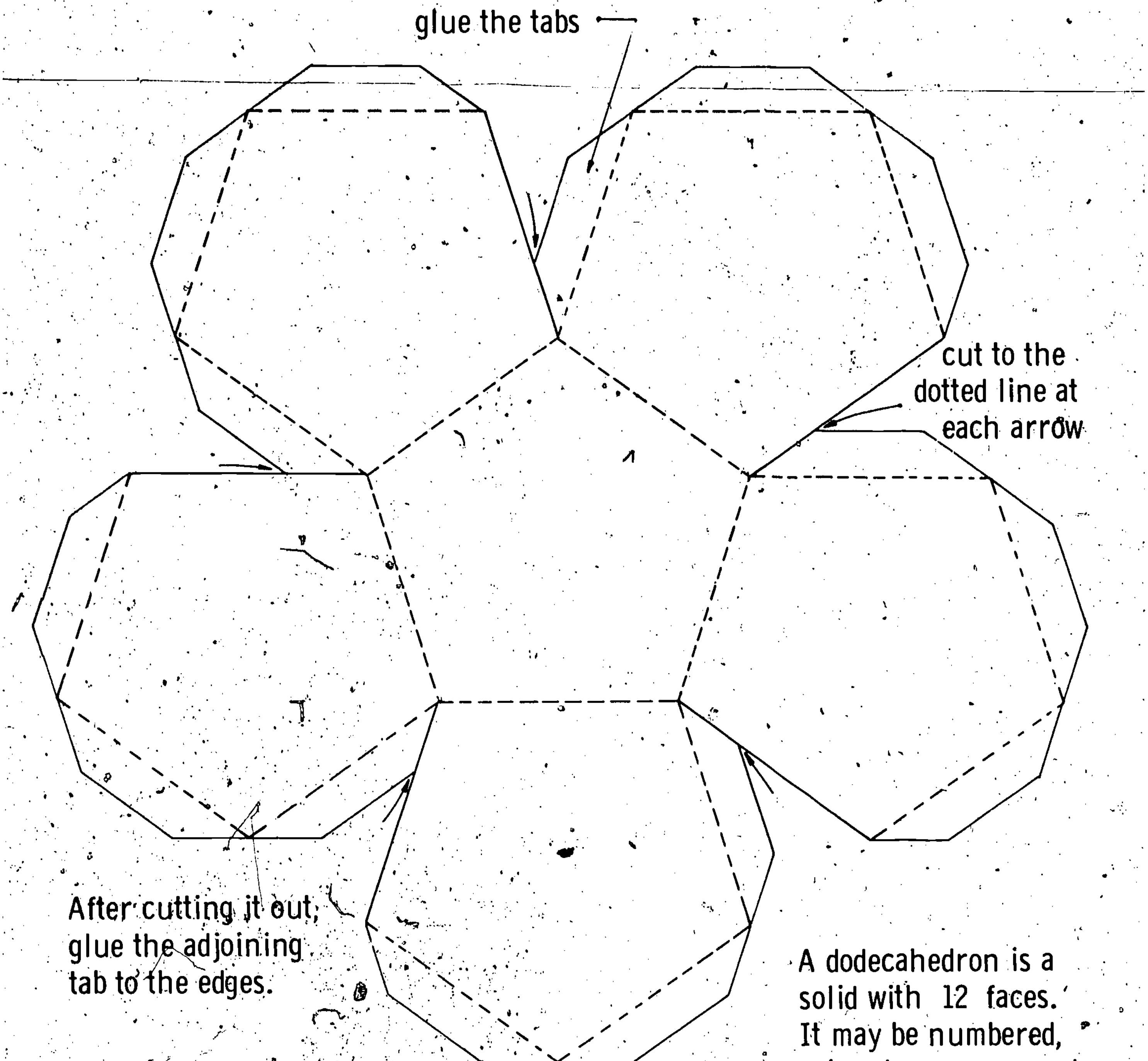
If the cube is to have "dots" on its faces, make the dots as shown below.

If the cube is to be numbered, use the numbers as shown so that opposite faces will have a sum of 7.

Cube
or
Hexahedron


glue tabs
The faces may be numbered from 1 to 8. Colors may also be used.

Octahedron

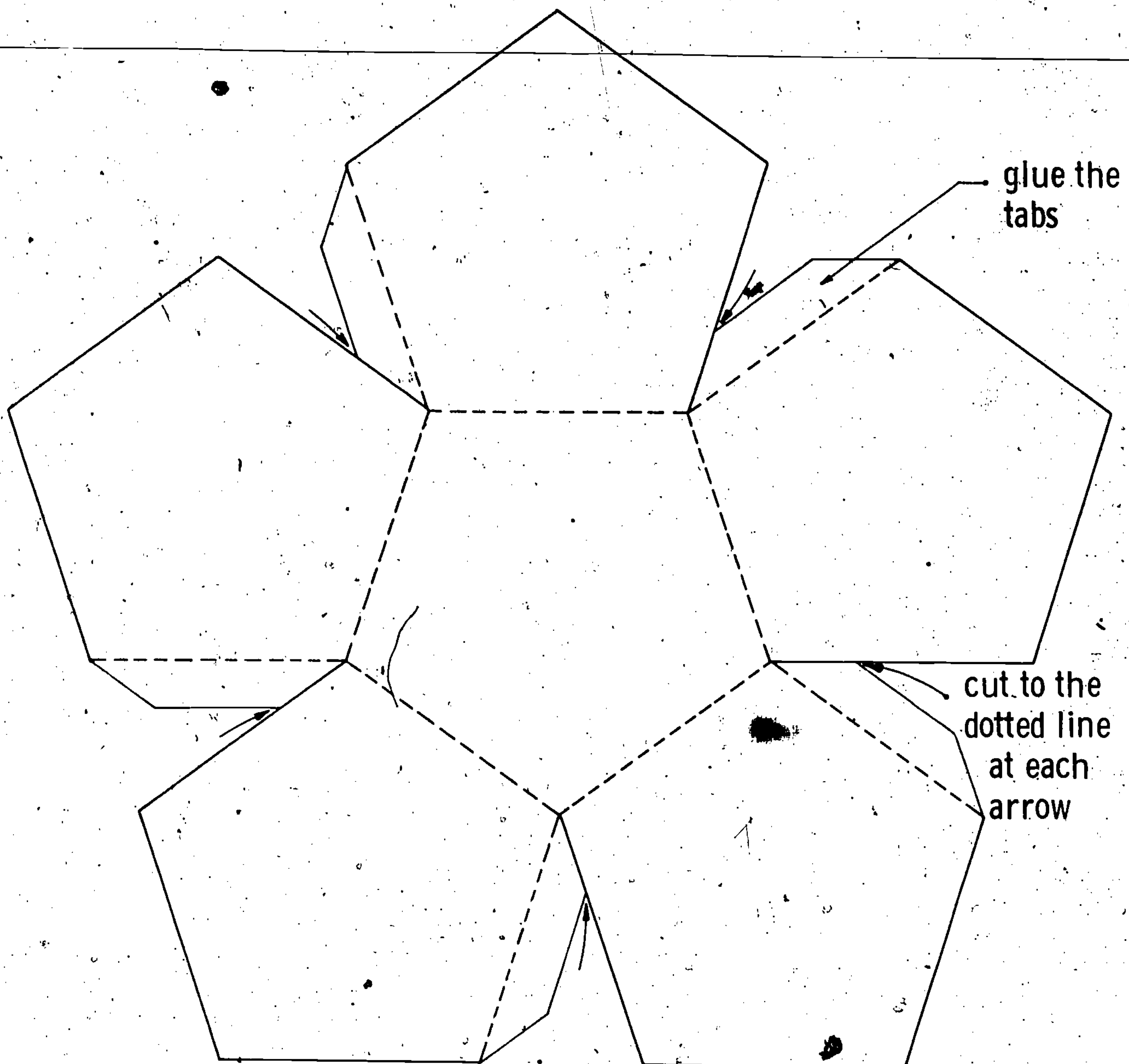


After cutting it out, glue the adjoining tab to the edges.

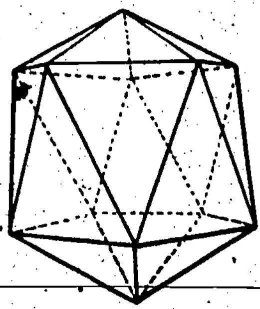
A dodecahedron is a solid with 12 faces. It may be numbered, colored, or even used as a calendar

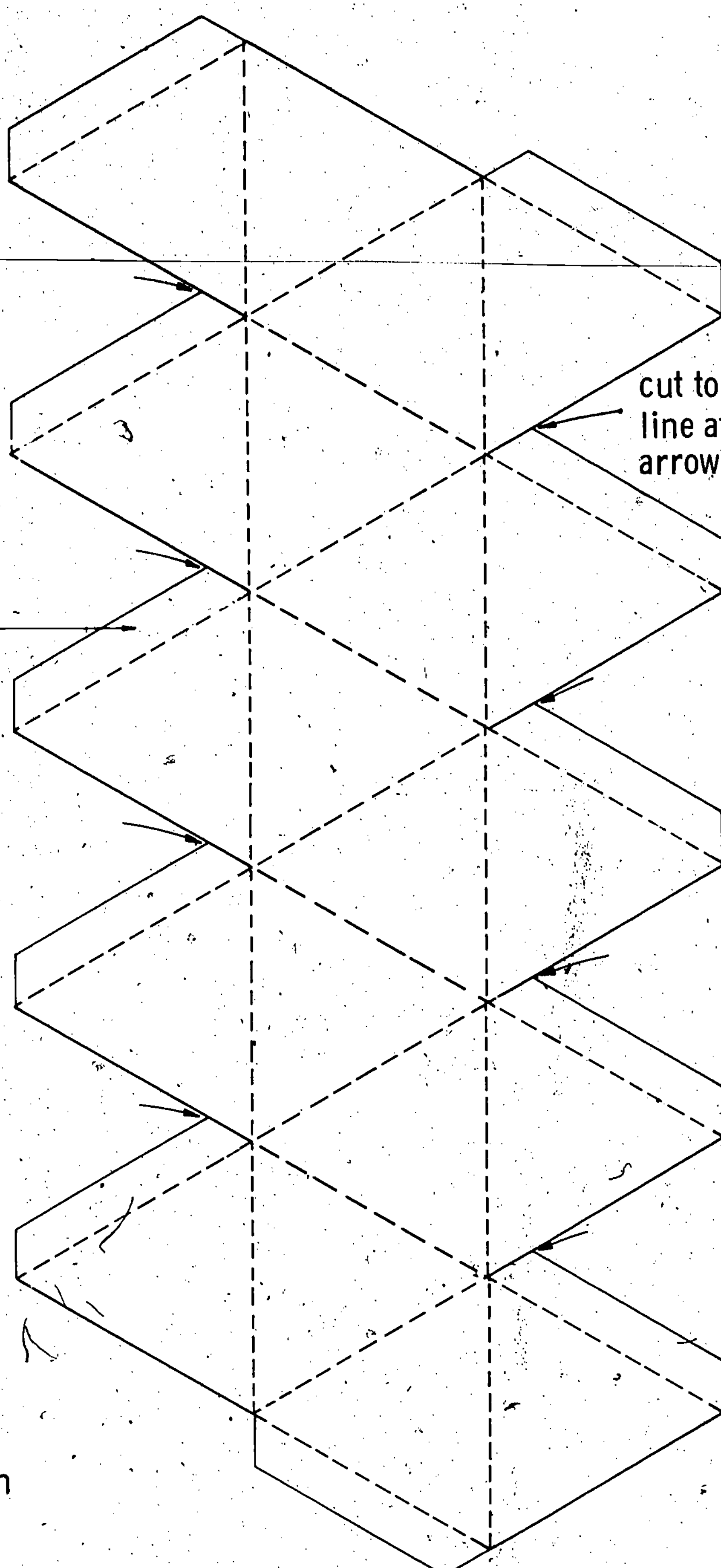
First half of a Dodecahedron

After cutting this out, do not glue adjoining tabs to the edge. Instead, glue one tab at a time to the first half of the dodecahedron.



Second half of a Dodecahedron





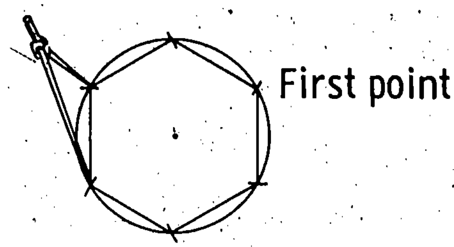
A solid with 20 faces

Icosahedron

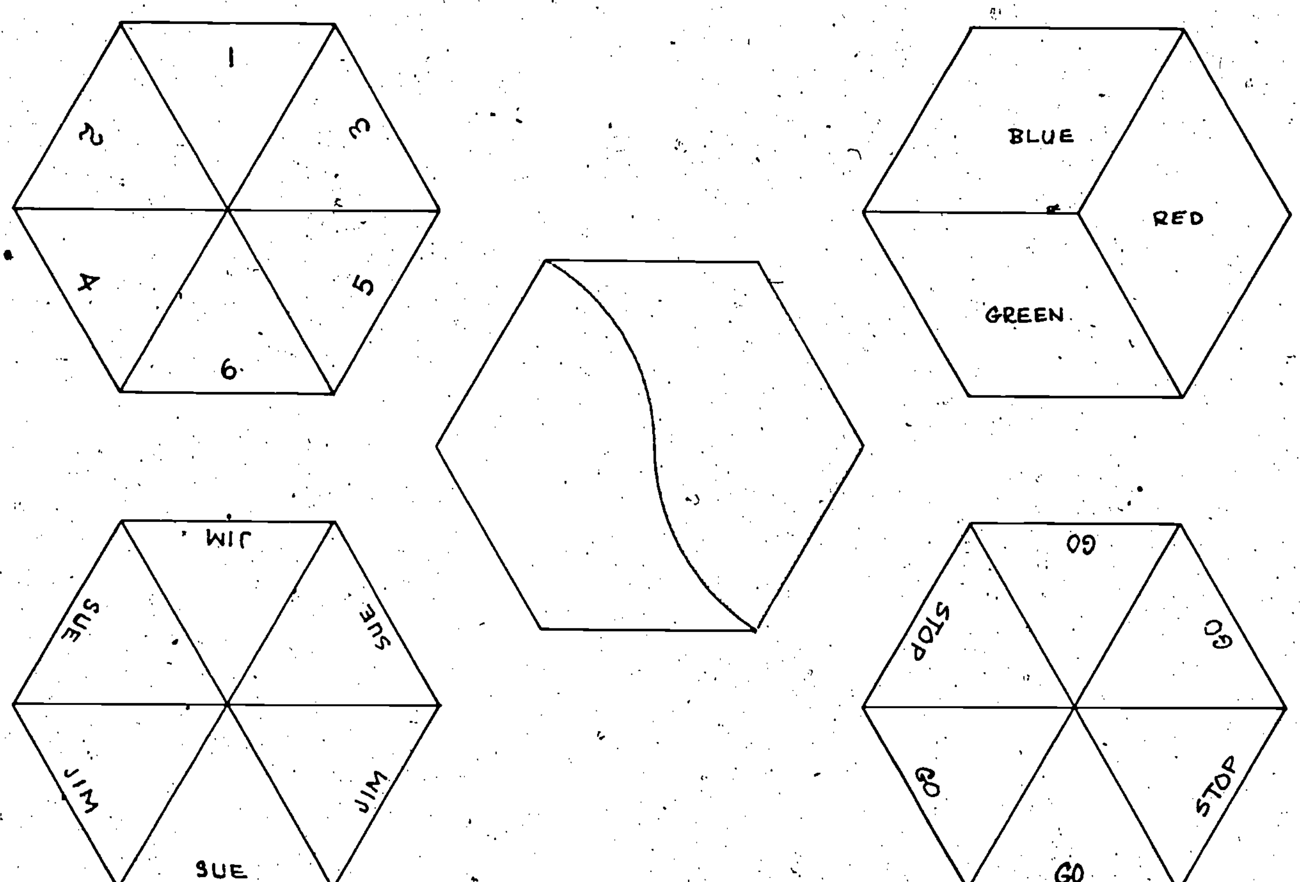
2. <u>Tops</u>

For a small top you need a piece of cardboard, a wood dowel or sucker stick, a straight edge, and a compass. Use a two-inch piece of dowel or sucker stick. Sharpen one end to a point with a pencil sharpener.

Use a compass and straight edge to make the dial from cardboard. Mark a point for the center and draw a circle with radius $1\frac{3}{4}$ inches. Mark any point on the circle and, with that point as center, strike an arc with the same radius to intersect the circle. Continue to strike arcs around the circle.



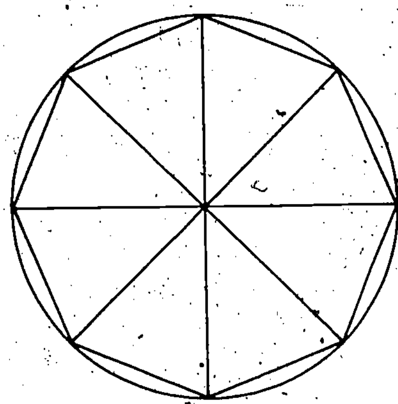
Connect points marked and cut off the outer part of the region to make a hexagonal region. Divide it as desired. Here are some suggestions:

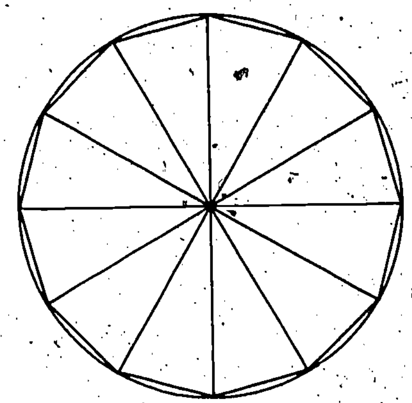


Poke the sharpened end of the stick through the center of the hexagon. (A drop of glue will help to keep it from wearing too large a hole, but experiment first to find the best balance for the top.) Spin the stick between thumb and forefinger. Spin the top on a flat surface. The edge that stops against the table is the one that is counted.

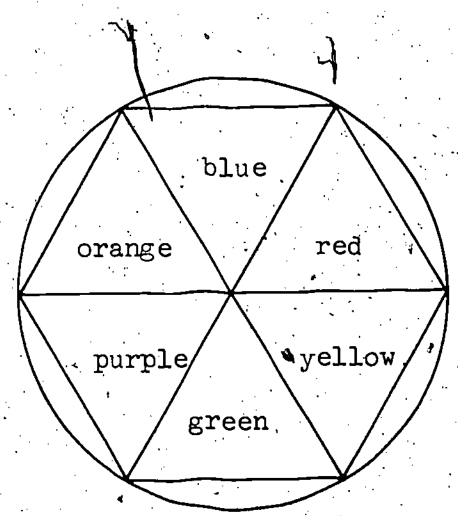
A large top may be made in the same way. An ordinary pencil makes an adequate stick. It is a good idea, however, to slit the cardboard along the division lines at the center before inserting the pencil. Use glue to fasten the cardboard to the pencil.

The top can also be made with eight edges or with twelve. Do you know how to do this?



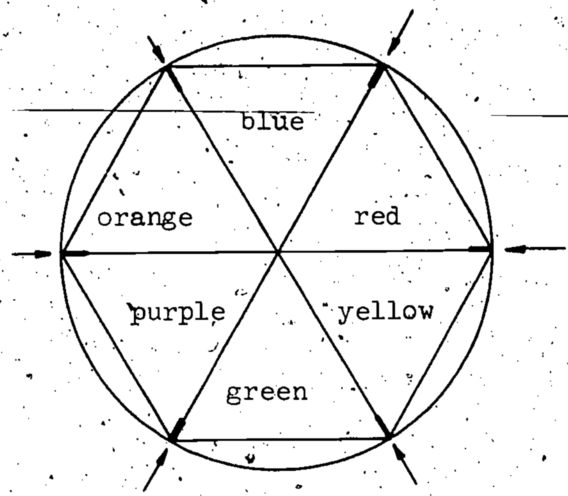


You can make a top with two or three dials. Use a sucker stick or thin dowel [illegible] inches long and sharpened at one end. Make a circular cardboard dial 6 inches in diameter. Inscribe a hexagon, but do not cut it out. Color as shown.



Insert the stick through the center and glue the dial to the stick.

When the glue is dry, place over the dial a "bearing" made of a piece of milk carton 1 inch square with a hole in the middle. Make another dial 6 inches in diameter, but if possible use cardboard that is slightly lighter in weight. Inscribe a hexagon, color, and cut as shown:

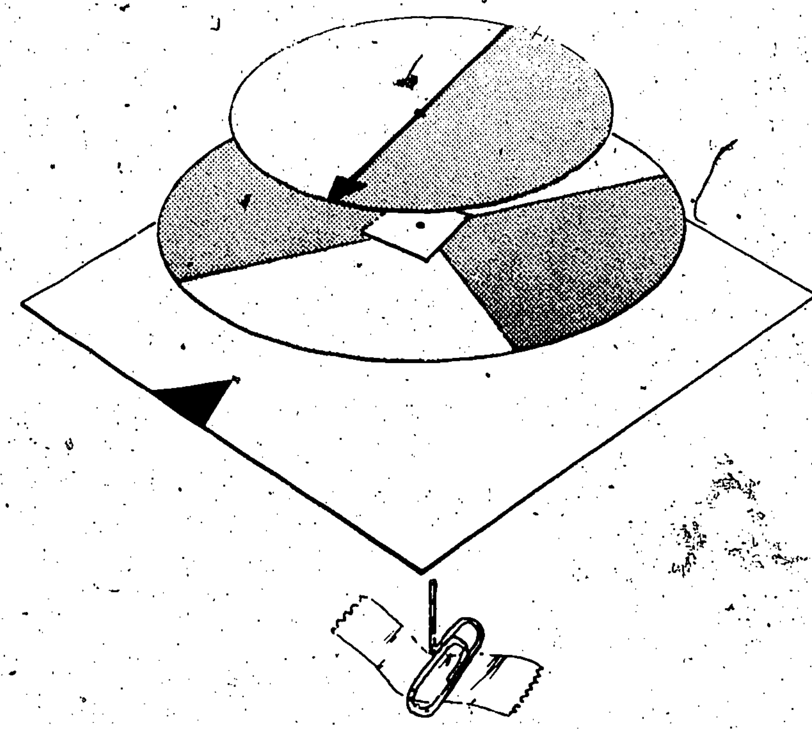


Make cuts $\frac{5}{8}$ inches long on the lines, as shown by arrows. Bend the cardboard up on the right of each cut to make a triangular "wind-catcher".

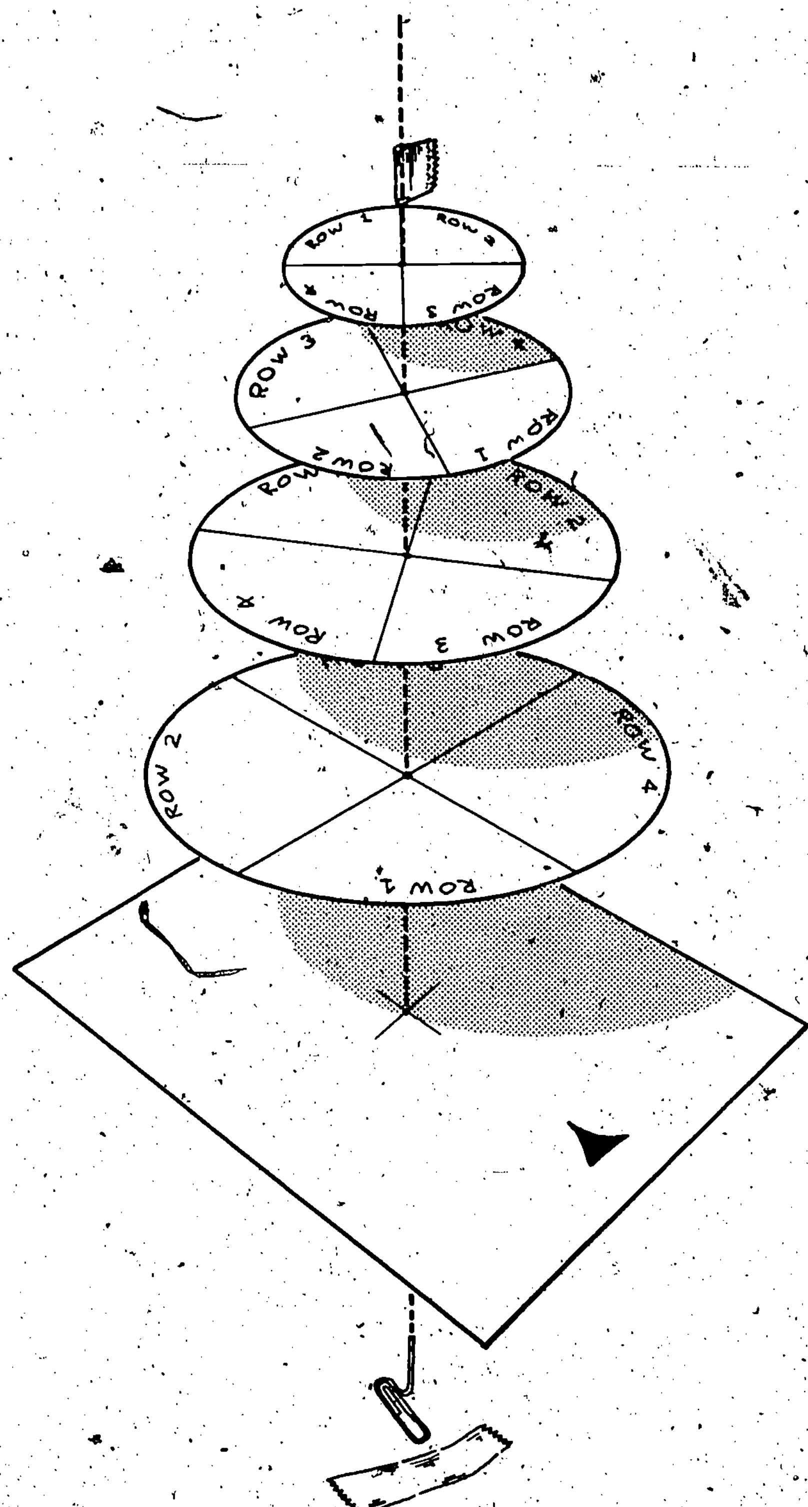
Place this dial on top of the bearing. Before spinning the top, line up the two dials so that the colors match. The bent-up cardboard will let you do this. Spin. Record the number of spins necessary before the colors again match. How many spins would you expect it to take?

3. <u>Spinners</u>

Spinners which come with most games have a fixed dial and an arrow which spins. You can make this type of spinner. Another spinner which is easily constructed is one in which the dial spins, as shown in the drawing. Construct the base out of heavy tag board. Use a heavy paper clip for the post and then various dials may be exchanged and used on the base.

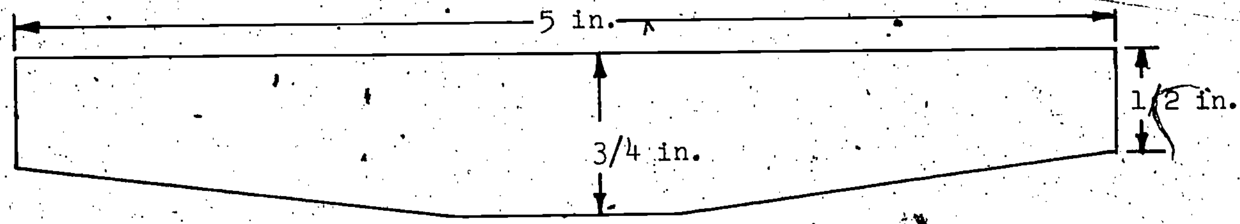


Another spinner with dials which go from large to small can illustrate how items can be put into an order. Each heavy cardboard dial is divided into a number of equal parts. The number of parts is the same as the number of items to be ordered. For example, to experiment to see how four rows in a classroom might be dismissed for lunch, each of four dials is divided into four equal parts. (A dial is made for each row.) Spin the dials and record how they line up with the arrow on the base. This device can be used to illustrate an orderly way of arriving at and listing the various arrangements.



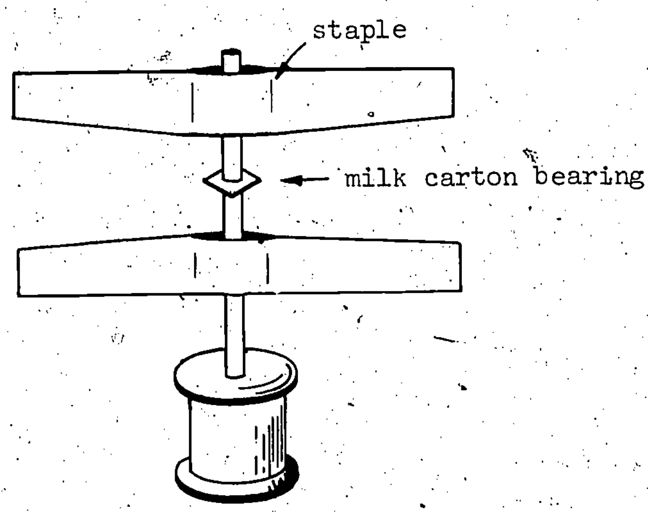
4. A Windmill

Cut four pieces of file card or tagboard to make two vanes.



A

Staple two pieces together in two places as shown in sketch B. Color one side of this vane red with a pencil or crayon. Insert a sucker stick in the middle between the two pieces. Use a punch to make a hole in a piece of milk carton one-half inch square for a "bearing". Staple the other two pieces of file card together to make a second vane. Mark one side as before, and insert the top of the stick between them. Paste a strip of gummed paper or tape over one end of a small spool. Put the end of the stick into the hole at the other end of the spool. Hold the spool and blow the file card vanes. They should turn quickly and independently.



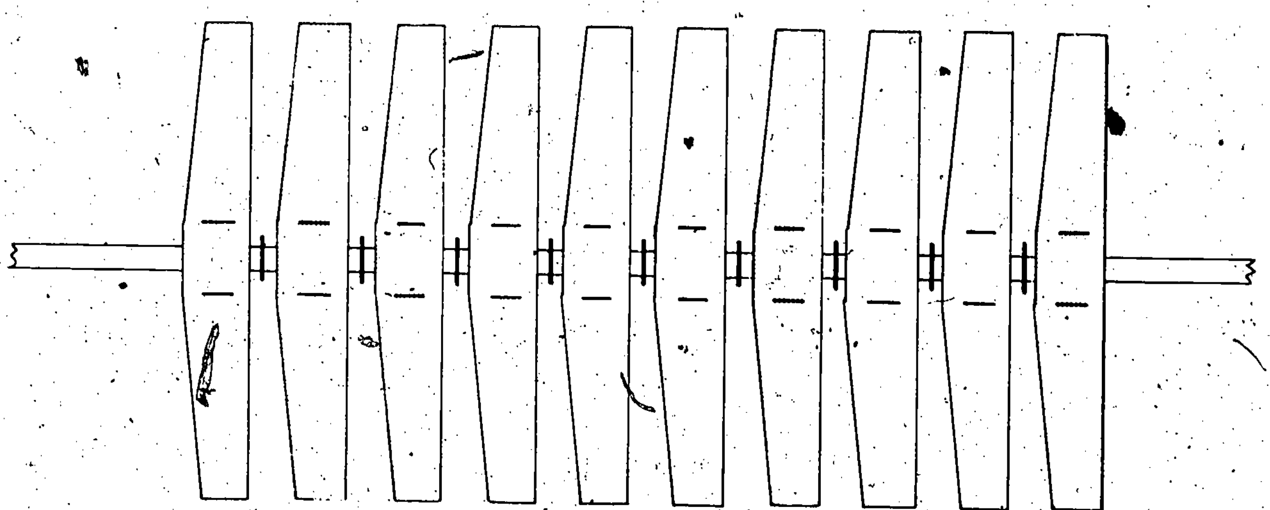
B

Practice blowing a few times. When vanes have stopped turning, lay them gently on a flat surface so that both vanes are flat on the surface. Then record whether both vanes are red, one is red and one is white, or both are white. After fifty trials, do the results of the experiment fit with the expected results?

Whirly-bird

This is another type of windmill.

Use a section of quarter-inch dowel about two feet long. Wax it by rubbing it with a piece of crayon. Follow the directions as given for a windmill and make as many file-card vanes and milk-carton bearings as desired. Hold the dowel at both ends, like a harmonica, and blow. When vanes have stopped turning, lay the Whirly-bird gently on a flat surface so that all vanes are on this surface. One vane corresponds to one coin, so ten vanes can be used to duplicate an experiment of "tossing ten coins". Can you think of other ways to color the vanes so that other experiments can be done?



5. Spoon device

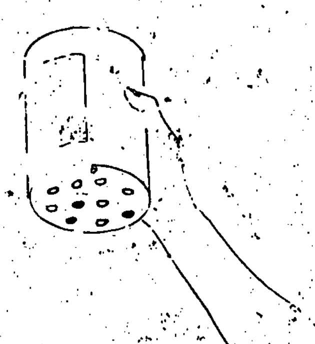
Use two plastic picnic spoons of different colors. Lay the handle of one on the handle of the other so that the bowls are at opposite ends and face opposite ways. Fasten with two rubber bands as shown at arrows.

Roll the handles between your palms and drop on a table from a height of a foot or so. The spoon on top counts. Is it just as likely that one spoon will be up as the other?

6. Sampling Boxes (Urns)

Many probability experiments require a sampling to be taken in a random manner. This device uses various colored marbles.

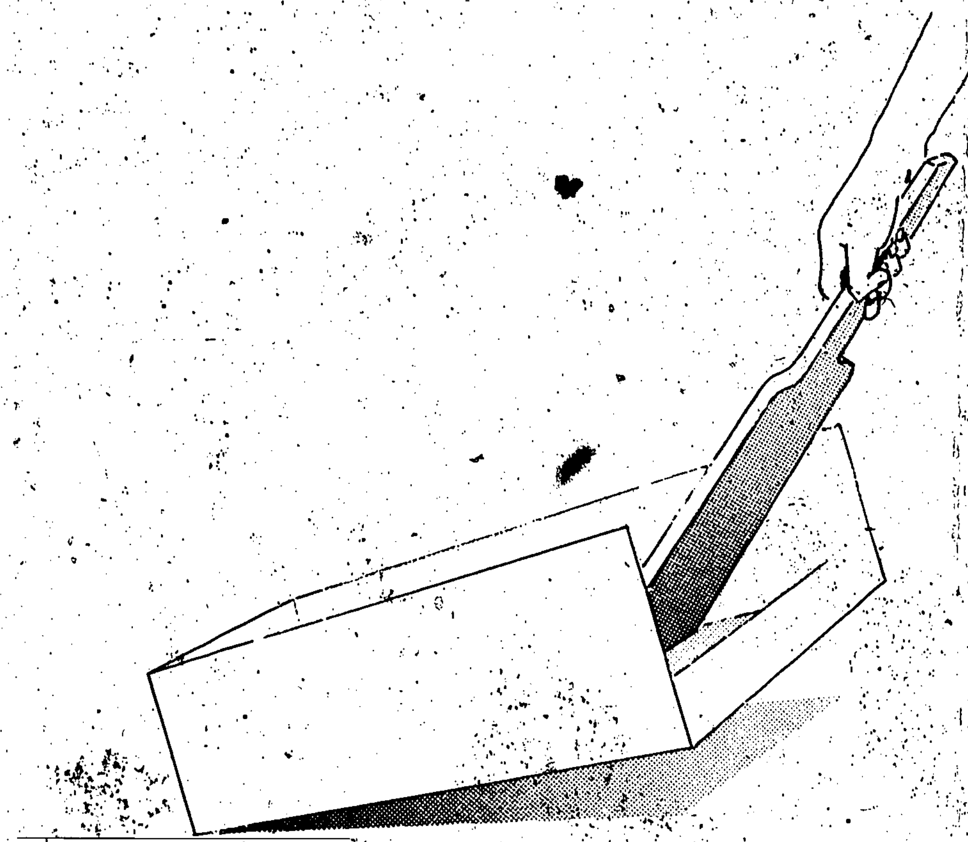
An oatmeal box serves well as the container (urn). Cut round holes, ten, for example, in the bottom of the box. These holes should have a diameter slightly smaller than the marbles so the marbles can be seen in them. Chinese Checker marbles serve well and come in packages of 6 colors, 10 of each.

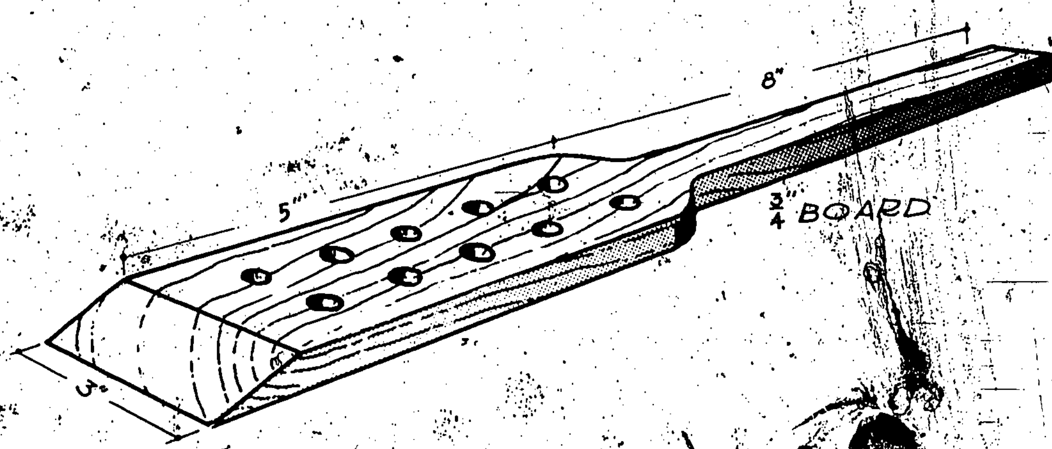


Example: Tell a friend you have 60 marbles in the container. Do not tell him how many there are of each color. Use, for example, 40 white and 20 black. Turn the urn up 10 times and record the number of black and white which show in each sampling. From the total, predict the probable number of white and black marbles in the urn.

A second sampling box can be made by using a box (rectangular or cylindrical) with a lid. It should be large enough to allow objects such as marbles or small balls to roll around. In a corner of a rectangular box or at the bottom edge of a cylindrical one, cut a hole just large enough to let one object go through easily. With the hole up, shake the box. Turn it over in your hand to let one object come out.

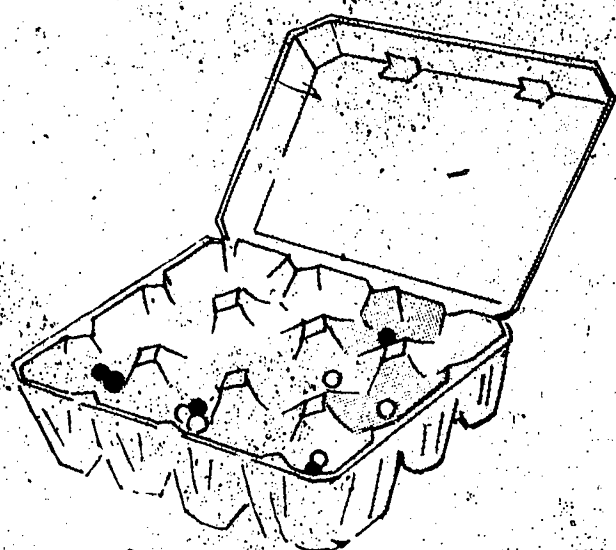
A third sampling device, a paddle, is made as shown in the drawing. The holes are slightly smaller in diameter than the marbles. Cut a box ~~as shown~~ (a shoe box works well). Place marbles in the box. Scoop with the paddle until all "holes" are filled. This gives a sample of the entire collection of marbles.




8"
5"
3"
3/4" BOARD

Egg-Carton Sampler

An egg carton and some marbles can be used for experiments. For example, color alternate pockets of the carton black. Place 5 black marbles and 5 white ones inside the carton. Close the lid, turn the carton upside down, and allow the marbles to roll around. Flip the carton upright and open the lid. Record the information you are interested in, for example, the number of black marbles in black pockets, etc.

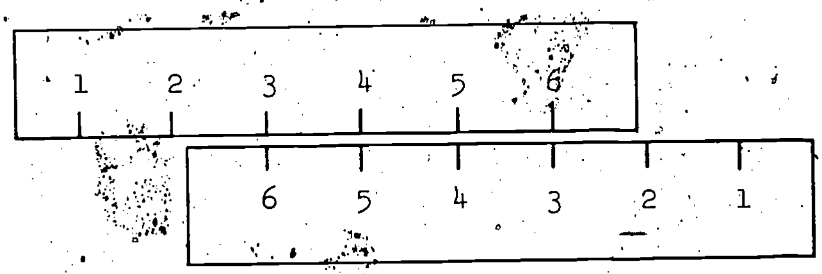


7. A Probability Scale

When tossing two dice or other regular solids, it helps to construct a table for counting how many ways a certain sum or product can be obtained. Two cubes, for example, give the following sums.

Sums	Possible Combinations						No. of Ways
2	(1,1)						1
3	(1,2),	(2,1)					2
4	(1,3),	(2,2),	(3,1)				3
5	(1,4),	(2,3),	(3,2),	(4,1)			4
6	(1,5),	(2,4),	(3,3),	(4,2),	(5,1)		5
7	(1,6),	(2,5),	(3,4),	(4,3),	(5,2),	(6,1)	6
8		(2,6),	(3,5),	(4,4),	(5,3),	(6,2)	5
9			(3,6),	(4,5),	(5,4),	(6,3)	4
10				(4,6),	(5,5),	(6,4)	3
11					(5,6),	(6,5)	2
12						(6,6)	1
						Total	36

This information can be placed on two number lines on strips of cardboard as in the sketch.

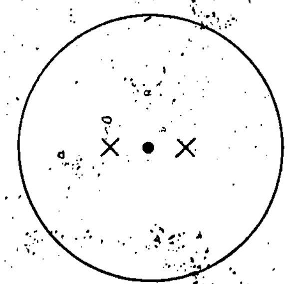


These two strips can be placed in pockets of a larger piece of cardboard so that by sliding the scales along, one quickly sees the number of possible combinations. This figure shows the 4 possible combinations of a sum of 9.

Scales for other solids may also be constructed.

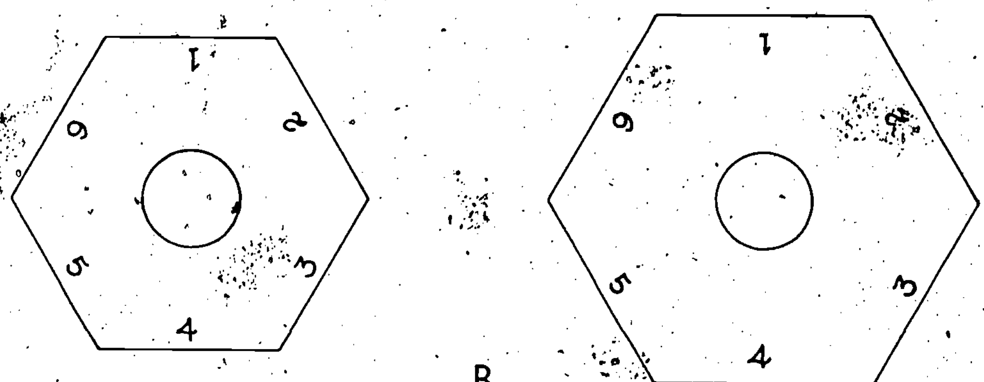
8. Hexawhirl

This gadget works like an old-fashioned button-on-a-string toy. Cut from cardboard two circles with radius $1\frac{3}{4}$ inches. In each, punch two holes just big enough to insert a piece of strong string. The holes should be punched on one of the diameters of the circle, each $\frac{1}{2}$-inch from the center. (See points marked × in Figure A.)



A

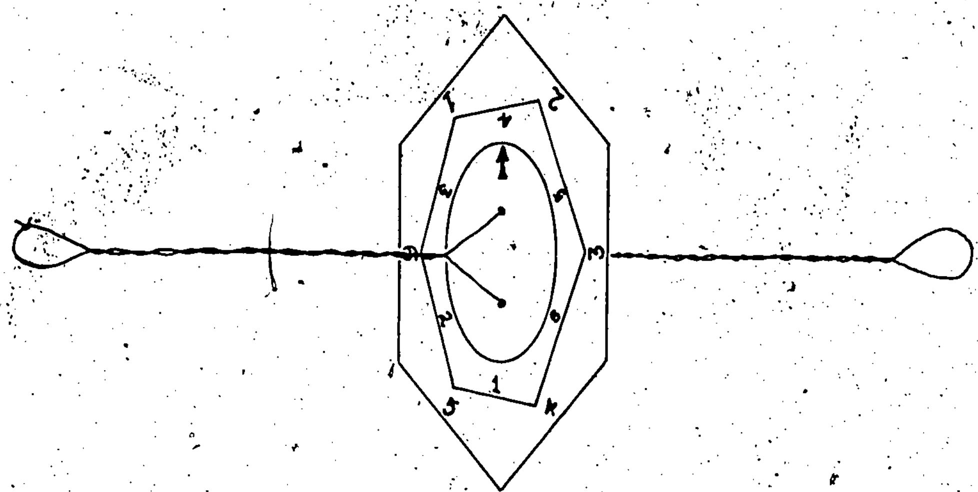
Make two or more cardboard hexagons using a $2\frac{1}{2}$-inch radius for the first, a 3-inch radius for the second, etc. Cut a hole with radius $1\frac{1}{4}$ inches in the middle of each hexagon and number the sides 1 through 6. (See Figure B.)



B

Use a strong string 50 inches long. Insert one end through a hole in one of the circles, through the smaller hexagon, then the larger hexagon (making sure the numbered sides face the same way), and then through the other circle. Leave a loop of string beyond the circle and insert the string through the other hole of the second circle, back through the larger hexagon, the smaller hexagon, and the first circle. Tie the ends of the string together to make a second loop. Adjust the cardboard pieces so that the loops on each side of the cardboard are the same length. There should be just enough space between the circular pieces for the hexagons to turn on the string. Fasten

the circles to the string with a drop of glue. Make an arrow on the circle next to the smaller hexagon. (See Figure C.)



C

To operate the hexawhirl, hold a loop in each hand and swing the cardboard pieces around and around (25 or more times) until the loops of string are twisted. Pull the loops until the twisting is undone, release to allow string to twist the other way, and pull again. With practice you can make the hexagons spin rapidly between the circles. Stop, and see which sides of the hexagons are in line with the arrow (2 on the smaller, 3 on the larger, for instance). Experiment to find out if the results are similar to those obtained by throwing two dice.